29. Limits

Exercise 29.1

1. Question

Show that $\lim_{x\to 0}\frac{x}{|x|}$ does not exist.

Answer

Given

f(x) = $\begin{cases}\frac{x}{x}, x > 0\\ \frac{x}{-x}, x < 0\end{cases}$

f(x) = $\begin{cases}1, x > 0\\ -1, x < 0\end{cases}$

To find $\lim_{x\to 0} f(x)$

To limit to exist, we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$......(2)

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h) = \lim_{h\to 0} 1 = 1$........(3)

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h) = \lim_{h\to 0} -1 = -1$......(4)

From above equations

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ (from 2)

Thus, limit does not exist.

2. Question

Find k so that $\lim_{x\to 2} f(x)$ may exist, where $f(x) = \begin{cases}2x+3, x \le 2\\ x+k,\ x > 2\end{cases}$.

Answer

Given f(x) = $\begin{cases}2x + 3, x \le 2\\ x + k, x > 2\end{cases}$

To find $\lim_{x\to 2} f(x)$

To limit to exist, we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

thus $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x)$

$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2 + h) = \lim_{h\to 0} 2(2 + h) + 3$

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2 - h) = \lim_{h\to 0} (2 - h) + k$

$\lim_{x\to 2} f(x) = f(2) = 2(2) + 3 = 7$

From (1)

$\lim_{h\to 0} 2(2 + h) + 3 = \lim_{h\to 0}(2 - h) + k$

2(2 + 0) + 3 = (2 - 0) + k

4 + 3 = 2 + k

5 = k

3. Question

Show that $\lim_{x\to 0}\frac{1}{x}$ does not exist.

Answer

$f(x) = \frac{1}{x}$

To find $\lim_{x\to 0} f(x)$

To limit to exist, we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus, to find the limit using the concept $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$......(2)

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h) = \lim_{h\to 0}\frac{1}{0+h} = \lim_{h\to 0}\frac{1}{h} = \infty$........(3)

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h) = \lim_{h\to 0}\frac{1}{0-h} = \lim_{h\to 0}\frac{-1}{h} = -\infty$......(4)

$\lim_{x\to 0} f(x) = f(0) = \frac{1}{0} = \infty$

From above equations

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x)$

Thus, limit does not exist.

4. Question

Let f(x) be a function defined by $f(x) = \begin{cases} \frac{3x}{|x| + 2x}, x \neq 0 \\ 0 \quad , x = 0 \end{cases}$.

Show that $f(x) =$ does not exist.

Answer

Given f(x) = $\begin{cases} \frac{3x}{|x| + 2x}, x \neq 0 \\ 0, x = 0 \end{cases}$

f(x) = $\begin{cases} \frac{3x}{x + 2x}, x > 0 \\ 0, x = 0 \\ \frac{3x}{-x + 2x} < 0 \end{cases}$

f(x) = $\begin{cases} 1, x > 0 \\ 0, x = 0 \\ 3 < 0 \end{cases}$

To find $\lim_{x\to 0} f(x)$

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$......(2)

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h) = \lim_{h\to 0} 1 = 1$........(3)

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h) = \lim_{h\to 0} 3 = 3$......(4)

$\lim_{x\to 0} f(x) = f(0) = 0$

From above equations

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0} f(x)$

Thus, limit does not exist.

5. Question

Let $f(x) = \begin{cases} x+1, \text{if } x > 0 \\ x-1, \text{if } x < 0 \end{cases}$. Prove that $\lim_{x\to 0} f(x)$ does not exist.

Answer

Given f(x) = $\begin{cases} x + 1, x > 0 \\ x - 1, x < 0 \end{cases}$

To find whether $\lim_{x\to 0} f(x)$ exists?

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to limit to exist $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$......(2)

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h) = \lim_{h\to 0} (0 + h) + 1 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h) = \lim_{h\to 0} (0 - h) - 1 = -1$

From above equations

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

Thus, the limit $\lim_{x\to 0} f(x)$ does not exists.

6. Question

Let $f(x) = \begin{cases} x+5, \text{if } x > 0 \\ x-4, \text{if } x < 0 \end{cases}$. Prove that $\lim_{x\to 0} f(x)$ does not exist.

Answer

Given f(x) = $\begin{cases} x + 5, x > 0 \\ x - 4, x < 0 \end{cases}$

To find whether $\lim_{x\to 0} f(x)$ exists?

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to limit to exist $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$......(2)

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h) = \lim_{h\to 0}(0 + h) + 5 = 5$

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h) = \lim_{h\to 0}(0 - h) - 4 = -4$

From above equations

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

Thus, the limit $\lim_{x\to 0} f(x)$ does not exists.

7. Question

Find $\lim_{x\to 3} f(x)$, where

$$f(x) = \begin{cases} 4, \text{if } x > 3 \\ x + 1, \text{if } x < 3 \end{cases}$$

Answer

Given f(x) = $\begin{cases} 4, x > 3 \\ x + 1, x < 3 \end{cases}$

To find $\lim_{x\to 3} f(x)$

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3} f(x)$......(2)

$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3 + h) = \lim_{h\to 0} 4 = 4$........(3)

$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3 - h) = \lim_{h\to 0}(3 - h) + 1 = 4$......(4)

From above equations

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

Thus from (2),(3) and (4)

$\mathbf{\lim_{x\to 3} f(x) = 4}$

8. Question

If $f(x) = \begin{cases} 2x + 3 & , x \le 0 \\ 3(x + 1), & x > 0 \end{cases}$. Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$.

Answer

Given f(x) = $\begin{cases} 2x + 3, x \le 0 \\ 3(x + 1), x > 0 \end{cases}$

(i)To find $\lim_{x\to 3} f(x)$

To limit to exist, we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$......(2)

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h) = \lim_{h\to 0} 3(0 + h + 1) = 3$........(3)

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} 2(0-h) + 3 = 3$......(4)

$\lim_{x\to 0} f(x) = f(0) = 2(0) + 3 = 3$.......(5)

From above equations

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$ thus the limit exists

Thus from (5)

$\lim_{x\to 0} f(x) = 3$

(ii) To find $\lim_{x\to 1} f(x)$

To limit to exist, we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x)$......(2)

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1 + h) = \lim_{h\to 0} 2(1 + h) + 3 = 5$........(3)

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 2(1-h) + 3 = 5$......(4)

From above equations

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$

Thus from (2),(3) and (4)

$\lim_{x\to 1} f(x) = 5$

9. Question

Find $\lim_{x\to 1} f(x)$, if $f(x) = \begin{cases} x^2 - 1 & ,x \le 1 \\ -x^2 - 1), x > 1 \end{cases}$.

Answer

Given f(x) = $\begin{cases} x^2 - 1, x \le 1 \\ -x^2 - 1, x > 1 \end{cases}$

To find $\lim_{x\to 1} f(x)$

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x)$......(2)

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1 + h) = \lim_{h\to 0} -(1 + h)^2 - 1 = \lim_{h\to 0} - 1^2 - h^2 - 2h - 1 = -2$ ....(3)

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} (1 + h)^2 - 1 = \lim_{h\to 0} 1^2 + h^2 + 2h - 1 = 0$......(4)

From above equations

$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$ thus the limit $\lim_{x\to 1} f(x)$ does not exists

10. Question

Evaluate $\lim_{x\to 0} f(x)$, where

$$f(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$$

Answer

Given f(x) = $\begin{cases}\frac{|x|}{x}, x \neq 0\\ 0, x = 0\end{cases}$

f(x) = $\begin{cases}\frac{x}{x}, x > 0\\ 0, x = 0\\ -\frac{x}{x} < 0\end{cases}$

f(x) = $\begin{cases}1, x > 0\\ 0, x = 0\\ -1 < 0\end{cases}$

To find $\lim_{x\to 0} f(x)$

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$......(2)

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} 1 = 1$........(3)

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} -1 = -1$......(4)

$\lim_{x\to 0} f(x) = f(0) = 0$

From above equations

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0} f(x)$

Thus limit does not exists

11. Question

Let $a_1, a_2, ...a_n$ be fixed real numbers such that f(x) = $(x - a_1)(x - a_2) ....(x - a_n)$

What is $\lim_{x\to a_1} f(x)$? For $a \neq a_1, a_2, ...,a_n$ compute $\lim_{x\to a} f(x)$

Answer

Given: f(x) = $(x - a_1)(x - a_2).....(x - a_n)$

$\lim_{x\to a_1} f(x) = (a_1 - a_1)(a_1 - a_2)\ldots..(a_1 - a_n)$

$\lim_{x\to a_1} f(x) = 0$

Now,

$\lim_{x\to a} f(x) = (a - a_1)(a - a_2)(a - a_3)\ldots.(a - a_n)$

12. Question

Find $\lim_{x\to 1^+} \frac{1}{x-1}$

Answer

Given f(x) = $\frac{1}{x-1}$

To find $\lim\limits_{x\to1^+} f(x)$

$$\lim_{x\to1^+} f(x) = \lim_{h\to0} f(1+h) = \lim_{h\to0}\frac{1}{(1+h)-1} = \lim_{h\to0}\frac{1}{h} = \frac{1}{0} = \infty$$

13 A. Question

Evaluate the following one - sided limits:

$$\lim_{x\to2^+}\frac{x-3}{x^2-4}$$

Answer

Given f(x) = $\frac{x-3}{x^2-4}$

To find $\lim\limits_{x\to2^+} f(x)$

$$\lim_{x\to2^+} f(x) = \lim_{h\to0} f(2+h) = \lim_{h\to0}\frac{(2+h)-3}{(2+h)^2-4} = \lim_{h\to0}\frac{h-1}{2^2+h^2+2h-4}$$
$$= \frac{0-1}{4+0^2+0-4} = -\frac{1}{0} = -\infty$$

13 B. Question

Evaluate the following one - sided limits:

$$\lim_{x\to2^-}\frac{x-3}{x^2-4}$$

Answer

Given f(x) = $\frac{x-3}{x^2-4}$

To find $\lim\limits_{x\to2^-} f(x)$

$$\lim_{x\to2^-} f(x) = \lim_{h\to0} f(2-h) = \lim_{h\to0}\frac{(2-h)-3}{(2-h)^2-4} = \lim_{h\to0}\frac{-h-1}{2^2+h^2-2h-4}$$
$$= \frac{0-1}{4+0^2-0-4} = -\frac{1}{0} = -\infty$$

13 C. Question

Evaluate the following one - sided limits:

$$\lim_{x\to0^+}\frac{1}{3x}$$

Answer

Given f(x) = $\frac{1}{3x}$

To find $\lim\limits_{x\to0^+} f(x)$

$$\lim_{x\to0^+} f(x) = \lim_{h\to0} f(0+h) = \lim_{h\to0}\frac{1}{3(0+h)} = \lim_{h\to0}\frac{1}{3h} = \frac{1}{0} = \infty$$

13 D. Question

Evaluate the following one - sided limits:

$$\lim_{x \to -8^+} \frac{2x}{x+8}$$

Answer

Given f(x) = $\frac{2x}{x+8}$

Factorizing f(x)

f(x) = $\frac{2x+16-16}{x+8}$

f(x) = $\frac{2(x+8)}{x+8} - \frac{16}{x+8}$

f(x) = $2 - \frac{16}{x+8}$

To find $\lim_{x \to -8^+} f(x)$

$$\lim_{x \to -8^+} f(x) = \lim_{h \to 0} f(-8 + h) = \lim_{h \to 0} 2 - \frac{16}{(-8 + h) + 8} = \lim_{h \to 0} 2 - \frac{16}{h} = 2 - \infty$$
$$= -\infty$$

13 E. Question

Evaluate the following one - sided limits:

$$\lim_{x \to 0^+} \frac{2}{x^{1/5}}$$

Answer

Given f(x) = $\frac{2}{x^{\frac{1}{5}}}$

To find $\lim_{x \to 0^+} f(x)$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{2}{(0 + h)^{\frac{1}{5}}} = \lim_{h \to 0} \frac{2}{h^{\frac{1}{5}}} = \frac{2}{0} = \infty$$

13 F. Question

Evaluate the following one - sided limits:

$$\lim_{x \to \frac{\pi}{2}^-} \tan x$$

Answer

Some standard limit are:

$$\lim_{x \to 0} (\tan x) \frac{1}{x} = 1$$

$$\lim_{x \to 0} (\sin x) \frac{1}{x} = 1$$

$$\lim_{x \to 0} (\cos x) = 1$$

Thus to find:

$$\lim_{x\to\frac{\pi}{2}^-}\tan x = \lim_{x\to\frac{\pi}{2}^-} f(x)$$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{h\to 0} f(\frac{\pi}{2}-h) = \lim_{h\to 0}\tan(\frac{\pi}{2}-h) = \lim_{h\to 0}\coth = \lim_{h\to 0}\frac{1}{\tanh} = \lim_{h\to 0}\frac{h}{h\tanh} = \lim_{h\to 0}\frac{1}{h} = \infty$$

13 G. Question

Evaluate the following one - sided limits:

$$\lim_{x\to-\frac{\pi}{2^+}} \sec x$$

Answer

Some standard limit are:

$$\lim_{x\to 0}(\tan x)\frac{1}{x} = 1$$

$$\lim_{x\to 0}(\sin x)\frac{1}{x} = 1$$

$$\lim_{x\to 0}(\cos x) = 1$$

Thus to find:

$$\lim_{x\to-\frac{\pi}{2}^+} \sec x = \lim_{x\to\frac{-\pi}{2}^-} f(x)$$

$$\lim_{x\to\frac{-\pi}{2}^+} f(x) = \lim_{h\to 0} f(-\frac{\pi}{2}+h) = \lim_{h\to 0}\sec(-\frac{\pi}{2}+h) = \lim_{h\to 0} -\text{cosech} = \lim_{h\to 0}\frac{-1}{\sinh} = \lim_{h\to 0}\frac{-h}{h\tanh} = \lim_{h\to 0} -\frac{1}{h} = -\infty$$

13 H. Question

Evaluate the following one - sided limits:

$$\lim_{x\to 0^-}\frac{x^2-3x+2}{x^3-2x^2}$$

Answer

Given $f(x) = \frac{x^2-3x+2}{x^3-2x^2}$

Factorizing f(x)

$f(x) = \frac{x^2-2x-x+2}{x^2(x-2)}$

$f(x) = \frac{x(x-2)-1(x-2)}{x^2(x-2)}$

$f(x) = \frac{(x-1)(x-2)}{x^2(x-2)}$

$f(x) = \frac{(x-1)}{x^2}$

To find $\lim_{x\to 0^-} f(x)$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{(0-h)-1}{(0-h)^2} = \lim_{h\to 0}\frac{-h-1}{h^2} = \frac{-1}{0} = -\infty$$

13 I. Question

Evaluate the following one - sided limits:

$$\lim_{x \to -2^+} \frac{x^2 - 1}{2x + 4}$$

Answer

$$\lim_{x \to -2^+} \frac{x^2 - 1}{2x + 4} = \lim_{h \to 0} \frac{[(-2 + h)^2 - 1]}{[2(-2 + h) + 4]} = \frac{h^2 - 4h + 3}{-4 + 2h + 4} = \infty$$

13 J. Question

Evaluate the following one - sided limits:

$$\lim_{x \to 0^-} (2 - \cot x)$$

Answer

Some standard limit are:

$$\lim_{x \to 0} (\tan x) \frac{1}{x} = 1$$

$$\lim_{x \to 0} (\sin x) \frac{1}{x} = 1$$

$$\lim_{x \to 0} (\cos x) = 1$$

Thus to find:

$$\lim_{x \to 0^-} 2 - \cot x = \lim_{x \to 0^-} f(x)$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) =$$

$$\lim_{h \to 0} 2 - \cot(0 - h) = \lim_{h \to 0} 2 - \cot(-h) == \lim_{h \to 0} 2 + \coth = \lim_{h \to 0} 2 + \frac{1}{\tanh} = \lim_{h \to 0} = 2 + \infty = \infty$$

13 K. Question

Evaluate the following one - sided limits:

(xi) $\lim_{x \to 0^-} 1 + \operatorname{cosec} x$

Answer

Some standard limit are:

$$\lim_{x \to 0} (\tan x) \frac{1}{x} = 1$$

$$\lim_{x \to 0} (\sin x) \frac{1}{x} = 1$$

$$\lim_{x \to 0} (\cos x) = 1$$

Thus to find:

$$\lim_{x \to 0^-} 1 + \operatorname{cosec} x = \lim_{x \to 0^-} f(x)$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) =$$

$$\lim_{h \to 0} 1 + \operatorname{cosec}(0 - h) = \lim_{h \to 0} 1 + \operatorname{cosec}(-h) == \lim_{h \to 0} 1 - \operatorname{cosech} = \lim_{h \to 0} 1 + \frac{-1}{\sinh} = 1 - \infty = -\infty$$

14. Question

Show that $\lim\limits_{x\to 0} e^{-1/x}$ does not exist.

Answer

Given f(x) = $e^{-\frac{1}{x}}$

To find $\lim\limits_{x\to 0} f(x)$

To limit to exist we know $\lim\limits_{x\to h^+} f(x) = \lim\limits_{x\to h^-} f(x) = \lim\limits_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0} f(x)$......(2)

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0 + h) = \lim\limits_{h\to 0} e^{-\frac{1}{0+h}} = \lim\limits_{h\to 0} e^{-\frac{1}{h}} = \frac{1}{e^{\frac{1}{0}}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$........(3)

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h) == \lim\limits_{h\to 0} e^{-\frac{1}{0-h}} = \lim\limits_{h\to 0} e^{-\frac{1}{-h}} = \lim\limits_{h\to 0} e^{\frac{1}{h}} = e^{\frac{1}{0}} = e^{\infty} = \infty$......(4)

$\lim\limits_{x\to 0} f(x) = f(0) = e^{-\frac{1}{0}} = \frac{1}{e^{\frac{1}{0}}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

From above equations

$\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0} f(x)$

Thus, limit does not exist.

15 A. Question

Find:

$\lim\limits_{x\to 2} [x]$

Answer

We know greatest integer [x] is the integer part.

For f(x) = [x]

To find:

$\lim\limits_{x\to 2} f(x)$

To limit to exist we know $\lim\limits_{x\to h^+} f(x) = \lim\limits_{x\to h^-} f(x) = \lim\limits_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2} f(x)$......(2)

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{h\to 0} f(2 + h) = \lim\limits_{h\to 0} [2 + h] = 2$........(3)

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{h\to 0} f(2-h) == \lim\limits_{h\to 0} [2-h] = 1$......(4)

$\lim\limits_{x\to 2} f(x) = f(2) = [2] = 2$

From above equations

$\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2} f(x)$

Thus, the limit does not exist.

15 B. Question

Find:

$$\lim_{x\to\frac{5}{2}} [x]$$

Answer

We know greatest integer [x] is the integer part.

For f(x) = [x]

To find:

$$\lim_{x\to 2.5} f(x)$$

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 2.5^+} f(x) = \lim_{x\to 2.5^-} f(x) = \lim_{x\to 2.5} f(x)$......(2)

$\lim_{x\to 2.5^+} f(x) = \lim_{h\to 0} f(2.5 + h) = \lim_{h\to 0} [2.5 + h] = 2$........(3)

$\lim_{x\to 2.5^-} f(x) = \lim_{h\to 0} f(2.5 - h) == \lim_{h\to 0} [2.5 - h] = 2$......(4)

$\lim_{x\to 2.5} f(x) = f(2.5) = [2.5] = 2$

From above equations

$\lim_{x\to 2.5^-} f(x) = \lim_{x\to 2.5^+} f(x) = \lim_{x\to 2.5} f(x)$

Thus, limit does exists.

15 C. Question

Find:

$$\lim_{x\to 1} [x]$$

Answer

We know greatest integer [x] is the integer part.

For f(x) = [x]

To find:

$$\lim_{x\to 1} f(x)$$

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x)$......(2)

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1 + h) = \lim_{h\to 0} [1 + h] = 1$........(3)

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1 - h) == \lim_{h\to 0} [1 - h] = 0$......(4)

$\lim_{x\to 1} f(x) = f(1) = [1] = 1$

From above equations

$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$

Thus limit does not exists.

16. Question

Prove that $\lim_{x \to a^+} [x] = [a]$ for all a ∈ R. Also, prove that $\lim_{x \to 1^-} [x] = 0$.

Answer

To Prove: $\lim_{x \to a^+} [x] = [a]$

L.H.S = $\lim_{x \to a^+} [x] = \lim_{h \to 0} [a + h] = [a]$ (Since, [a + h] = [a])

Hence, Proved.

Also,

To prove: $\lim_{x \to 1^-} [x] = 0$

L.H.S = $\lim_{x \to 1^-} [x] = \lim_{h \to 0} [1 - h] = 0$ (Since, [1 – h] = 0)

Hence, Proved.

17. Question

Show that $\lim_{x \to 2^-} \frac{x}{[x]} \neq \lim_{x \to 2^+} \frac{x}{[x]}$.

Answer

We know greatest integer [x] is the integer part.

For f(x) = x/[x]

To show

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

Proof:

To limit to exist we know $\lim_{x \to h^+} f(x) = \lim_{x \to h^-} f(x) = \lim_{x \to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x)$......(2)

$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} \frac{2 + h}{[2 + h]} = \frac{2 + 0}{2} = 1$........(3)

$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h) = = \lim_{h \to 0} \frac{2 - h}{[2 - h]} = \frac{2}{1} = 2$......(4)

From above equations

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

18. Question

Find $\lim_{x \to 3^+} \frac{x}{[x]}$. Is it equal to $\lim_{x \to 3^-} \frac{x}{[x]}$.

Answer

We know greatest integer [x] is the integer part.

For f(x) = x/[x]

To show

$\lim_{x\to3^-} f(x) \neq \lim_{x\to3^+} f(x)$

Proof:

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to3^+} f(x) = \lim_{x\to3^-} f(x) = \lim_{x\to3} f(x)$......(2)

$\lim_{x\to3^+} f(x) = \lim_{h\to0} f(3+h) = \lim_{h\to0} \frac{3+h}{[3+h]} = \frac{3+0}{3} = 1$........(3)

$\lim_{x\to3^-} f(x) = \lim_{h\to0} f(3-h) == \lim_{h\to0} \frac{3-h}{[3-h]} = \frac{3-0}{2} = \frac{3}{2}$......(4)

From above equations

$\lim_{x\to3^-} f(x) \neq \lim_{x\to3^+} f(x)$

19. Question

Find $\lim_{x\to-5/2} [x]$.

Answer

We know greatest integer [x] is the smallest integer nearest to that number .

For f(x) = [x]

To find:

$\lim_{x\to-2.5} f(x)$

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to-2.5^+} f(x) = \lim_{x\to-2.5^-} f(x) = \lim_{x\to-2.5} f(x)$......(2)

$\lim_{x\to-2.5^+} f(x) = \lim_{h\to0} f(-2.5+h) = \lim_{h\to0} [-2.5+h] = -3$........(3)

$\lim_{x\to-2.5^-} f(x) = \lim_{h\to0} f(-2.5-h) == \lim_{h\to0} [-2.5-h] = -3$......(4)

$\lim_{x\to-2.5} f(x) = f(2.5) = [-2.5] = -3$

From above equations

$\lim_{x\to-2.5^-} f(x) = \lim_{x\to-2.5^+} f(x) = \lim_{x\to-2.5} f(x)$

Thus limit does exists

20. Question

Evaluate $\lim_{x\to2} f(x)$ (if it exists), where $f(x) = \begin{cases} x-[x], x<2 \\ 4 \quad , x=2 \\ 3x-5, x>2 \end{cases}$.

Answer

Given $f(x) = \begin{cases} x-[x], x<2 \\ 4, x = 2 \\ 3x-5, x>2 \end{cases}$

To find $\lim_{x\to3} f(x)$

To limit to exist we know $\lim_{x\to h^+} f(x) = \lim_{x\to h^-} f(x) = \lim_{x\to h} f(x)$ ....(1)

Thus to find the limit using the concept $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x)$......(2)

$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} 3(2+h)-5 = 6+0-5 = 1$........(3)

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} 2-h+[2-h] = 2-h+1 = 3$......(4)

$\lim_{x\to 2} f(x) = f(2) = 4$.....(5)

From above equations

$\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x) \neq \lim_{x\to 2} f(x)$

Thus the limit does not exist

21. Question

Show that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Answer

To Prove: $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist

Let us take the left-hand limit for the function:

$\text{L.H.L} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \sin\left(\frac{1}{0-h}\right) = -\lim_{h\to 0} \sin\left(\frac{1}{h}\right)$

Now, multiplying and dividing by h, we get,

$\text{L.H.L} = -\frac{\lim_{h\to 0} \sin\left(\frac{1}{h}\right)}{\frac{1}{h}} \times \frac{1}{h} = -1 \times \frac{1}{0} = -\infty$

Now, taking the right-hand limit of the function, we get,

$\text{R.H.L} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \sin\left(\frac{1}{0+h}\right) = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$

Now, multiplying and dividing by h, we get,

$\text{R.H.L} = \frac{\lim_{h\to 0} \sin\left(\frac{1}{h}\right)}{\frac{1}{h}} \times \frac{1}{h} = 1 \times \frac{1}{0} = \infty$

Clearly, L.H.L ≠ R.H.L

Hence, limit does not exist.

22. Question

Let $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, \text{where } x \neq \frac{\pi}{2} \\ 3 \quad, \text{where } x \neq \frac{\pi}{2} \end{cases}$ and if $\lim_{x\to \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$, find the value of k.

Answer

$f(x) = \begin{Bmatrix} \frac{k\cos x}{\pi - 2x}, \text{where } x \neq \frac{\pi}{2} \\ 3, \text{where } x \neq \frac{\pi}{2} \end{Bmatrix}$

Let us find the limit of the function at $x = \frac{\pi}{2}$.

Let $y = x - \frac{\pi}{2}, \pi - 2x = -2y$

Therefore,

L.H.L $= \lim_{y \to 0^-} \frac{k\cos x}{\pi - 2x} = \lim_{h \to 0} \frac{\left(k\cos\left(y + \frac{\pi}{2}\right)\right)}{-2y} = \lim_{y \to 0} \frac{-k\sin y}{-2y} = \frac{k}{2}$

Now, $\frac{k}{2} = 3$

Hence, k = 6.

Exercise 29.2

1. Question

Evaluate the following limits:

$$\lim_{x \to 1} \frac{x^2 + 1}{x + 1}$$

Answer

Given limit $\Rightarrow \lim_{x \to 1} \frac{x^2 + 1}{x + 1}$

Putting the value of limits directly, i.e., x = 1, we have

$$\Rightarrow \frac{1^2 + 1}{1 + 1}$$

$$\Rightarrow \frac{2}{2}$$

$$\Rightarrow 1$$

Hence the value of the given limit is 1.

2. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2}$$

Answer

Given limit $\Rightarrow \lim_{x \to 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2}$

Putting the value of limits directly, i.e. x = 0, we have

$$\Rightarrow \frac{2(0^2) + 3(0) + 4}{0^2 + 3(0) + 2}$$

$$\Rightarrow \frac{4}{2}$$

$$\Rightarrow 2$$

Hence the value of the given limit is 2.

3. Question

Evaluate the following limits:

$$\lim_{x \to 3} \frac{\sqrt{2x+3}}{x+3}$$

Answer

Given limit $\Rightarrow \lim_{x \to 3} \frac{\sqrt{2x+3}}{x+3}$

Putting the value of limits directly, i.e. x = 0, we have

$$\Rightarrow \frac{\sqrt{2(3)+3}}{3+3}$$

$$\Rightarrow \frac{\sqrt{3}}{3}$$

$$\Rightarrow \frac{3}{6}$$

$$\Rightarrow \frac{1}{2}$$

Hence the value of the given limit is 0.5

4. Question

Evaluate the following limits:

$$\lim_{x \to 1} \frac{\sqrt{x+8}}{\sqrt{x}}$$

Answer

Given limit $\Rightarrow \lim_{x \to 1} \frac{\sqrt{x+8}}{x}$

Putting the values of limits directly, i.e. x = 1, we have

$$\Rightarrow \frac{\sqrt{1+8}}{1}$$

$$\Rightarrow \frac{\sqrt{9}}{1}$$

$$\Rightarrow 3$$

Hence the value of the given limit is 3.

5. Question

Evaluate the following limits:

$$\lim_{x \to a} \frac{\sqrt{x}+\sqrt{a}}{x+a}$$

Answer

Given limit $\Rightarrow \lim_{x \to a} \frac{\sqrt{x}+\sqrt{a}}{x+a}$

Putting the values of limit directly, i.e. x = a, we have

$$\Rightarrow \frac{\sqrt{a}+\sqrt{a}}{a+a}$$

$$\Rightarrow \frac{2\sqrt{a}}{2a}$$

$\Rightarrow \frac{1}{\sqrt{a}}$

Hence the value of the given limit is $\Rightarrow \frac{1}{\sqrt{a}}$

6. Question

Evaluate the following limits:

$$\lim_{x \to 1} \frac{1+(x-1)^2}{1+x^2}$$

Answer

Given limit $\Rightarrow \lim_{x \to 1} \frac{1+(x-1)^2}{1+x^2}$

Putting the values of limits directly, i.e. x = 1, we have

$\Rightarrow \frac{1+(1-1)^2}{1+1^2}$

$\Rightarrow \frac{1}{2}$

Hence the value of the given limit is 0.5

7. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{x^{2/3}-9}{x-27}$$

Answer

Given limit $\Rightarrow \lim_{x \to 0} \frac{x^{2/3}-9}{x-27}$

Putting the value of limit directly, i.e. x = 0, we have

$\Rightarrow \frac{0^{2/3}-9}{0-27}$

$\Rightarrow \frac{-9}{-27}$

$\Rightarrow \frac{1}{3}$

Hence the value of the given limit is $\Rightarrow \frac{1}{3}$

8. Question

Evaluate the following limits:

$$\lim_{x \to 0} 9$$

Answer

Given the limit $\Rightarrow \lim_{x \to 0} 9$

Always remember the limiting value of a constant (such as 4, 13, b, etc.) is the constant itself.

So, the limiting value of constant 9 is itself, i.e., 9.

9. Question

Evaluate the following limits:

$$\lim_{x \to 2} (3 - x)$$

Answer

Given the limit ⇒ $\lim_{x \to 2}(3 - x)$

Putting the limiting value directly, i.e. x = 2, we have

⇒ $(3 - 2)$

⇒ 1

Hence the value of the given limit is 1.

10. Question

Evaluate the following limits:

$$\lim_{x \to -1} (4x^2 + 2)$$

Answer

Given limit ⇒ $\lim_{x \to -1}(4x^2 + 2)$

Putting the value of limits directly, we have

⇒ $(4(-1)^2 + 2)$

⇒ $(4(1) + 2)$

⇒ 6

Hence the value of the given limit is 6.

11. Question

Evaluate the following limits:

$$\lim_{x \to -1} \frac{x^3 - 3x + 1}{x - 1}$$

Answer

Given the limit ⇒ $\lim_{x \to -1} \frac{x^3 - 3x + 1}{x - 1}$

Putting the value of limits directly, i.e. x = -1, we have

⇒ $\frac{(-1)^3 - 3(-1) + 1}{(-1) - 1}$

⇒ $\frac{-1 + 3 + 1}{-2}$

⇒ $\frac{-3}{2}$

Hence the value of the given limit is ⇒ $\frac{-3}{2}$

12. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{3x + 1}{x + 3}$$

Answer

Given limit $\Rightarrow \lim\limits_{x \to 0} \frac{3x+1}{x+3}$

Putting the value of limit directly, i.e. x = 0, we have

$\Rightarrow \frac{3(0)+1}{0+3}$

$\Rightarrow \frac{1}{3}$

Hence the value of the given limit is $\Rightarrow \frac{1}{3}$

13. Question

Evaluate the following limits:

$$\lim_{x \to 3} \frac{x^2 - 9}{x + 2}$$

Answer

Given limit $\Rightarrow \frac{1}{3}$

Putting the value of limits directly, i.e. x = 3, we have

$\Rightarrow \frac{3^2 - 9}{3 + 2}$

$\Rightarrow \frac{0}{5}$

$\Rightarrow 0$

Hence the value of the given limit is 0.

14. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{ax + b}{cx + d}, d \neq 0$$

Answer

Given limit $\Rightarrow \frac{1}{3}$

Putting the value of limits directly, i.e. x = 0, we have

$\Rightarrow \lim\limits_{x \to 0} \frac{ax + b}{cx + d}$

$\Rightarrow \frac{b}{d}$

The given condition d ≠ 0 is reasonable because the denominator cannot be zero.

Hence the value of the given limit is $\frac{b}{d}$.

Exercise 29.3

1. Question

Evaluate the following limits:

$$\lim_{x \to -5} \frac{2x^2 + 9x - 5}{x + 5}$$

Answer

$$= \frac{2(-5)^2 + 9(-5) - 5}{(-5) + 5}$$

$$= \frac{50 - 50}{(-5) + 5}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \to -5} \frac{2x^2 + 9x - 5}{x + 5}$$

$$= \lim_{x \to -5} \frac{2x^2 + 10x - x - 5}{x + 5}$$

$$= \lim_{x \to -5} \frac{2x(x + 5) - (x + 5)}{x + 5}$$

$$= \lim_{x \to -5} \frac{(2x - 1)(x + 5)}{x + 5}$$

$$= \lim_{x \to -5} 2x - 1$$

= 2(-5)-1

= -11

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \to -5} \frac{d(2x^2 + 9x - 5)}{d(x + 5)}$$

$$= \lim_{x \to -5} \frac{4x + 9}{1}$$

= 4(-5) + 9

= -11

2. Question

Evaluate the following limits:

$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3}$$

Answer

$$= \frac{(3)^2 - 4(3) + 3}{(3)^2 - 2(3) - 3}$$

$$= \frac{12 - 12}{(-9) + 9}$$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 3} \frac{(x^2 - 4x + 3)}{(x^2 - 2x - 3)}$

$= \lim_{x \to 3} \frac{(x^2 - 3x - x + 3)}{(x^2 - 3x + x - 3)}$

$= \lim_{x \to 3} \frac{x(x - 3) - 1(x - 3)}{x(x - 3) + 1(x - 3)}$

$= \lim_{x \to 3} \frac{(x - 3)(x - 1)}{(x - 3)(x + 1)}$

$= \lim_{x \to 3} \frac{(x - 1)}{(x + 1)}$

$= \frac{(3 - 1)}{(3 + 1)}$

$= \frac{2}{4} = \frac{1}{2}$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to 3} \frac{d(x^2 - 4x + 3)}{d(x^2 - 2x - 3)}$

$= \lim_{x \to 3} \frac{2x - 4}{2x - 2}$

$= \frac{2(3) - 4}{2(3) - 2}$

$= \frac{2}{4} = \frac{1}{2}$

3. Question

Evaluate the following limits:

$\lim_{x \to 3} \frac{x^4 - 81}{x^2 - 9}$

Answer

$= \frac{(3)^4 - 81}{(3)^2 - 9}$

$= \frac{81 - 81}{(-9) + 9}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 3} \frac{(x^4 - 81)}{(x^2 - 9)}$

$= \lim_{x\to 3} \frac{(x^4 - 3^4)}{(x^2 - 3^2)}$

$= \lim_{x\to 3} \frac{((x^2)^2 - (3^2)^2)}{(x^2 - 3^2)}$

Since a^2-b^2 = (a + b)(a-b)

Thus

$= \lim_{x\to 3} \frac{(x^2 - 3^2)(x^2 + 3^2)}{(x^2 - 3^2)}$

$= \lim_{x\to 3}(x^2 + 3^2)$

= $3^2 + 3^2$

= 18

Method 2:

By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x\to 3} \frac{d(x^4-81)}{d(x^2-9)}$

$= \lim_{x\to 3} \frac{4x^3}{2x}$

$= \frac{4(3)^3}{2}$

= 54

4. Question

Evaluate the following limits: $\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4}$

Answer

$= \frac{(2)^3-8}{(2)^2-4}$

$= \frac{8-8}{(4)-4}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x\to 2} \frac{(x^3 - 8)}{(x^2 - 4)}$

$= \lim_{x\to 2} \frac{(x^3 - 2^3)}{(x^2 - 2^2)}$

$= \lim_{x\to 2} \frac{(x - 2)(x^2 + 2^2 + 2x)}{(x + 2)(x - 2)}$

Since a^3-b^3 = (a-b)(a^2 + b^2 + ab) & a^2-b^2 = (a + b)(a-b)

$= \lim\limits_{x \to 2} \frac{(x^2 + 2^2 + 2x)}{(x + 2)}$

$= \frac{(2^2 + 2^2 + 2(2))}{(2 + 2)}$

$= \frac{3.4}{(4)}$

= 3

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim\limits_{x \to 2} \frac{d(x^3 - 8)}{d(x^2 - 4)}$

$= \lim\limits_{x \to 2} \frac{3x^2}{2x}$

$= \lim\limits_{x \to 2} \frac{3x}{2}$

$= \frac{3(2)}{2}$

= 3

5. Question

Evaluate the following limits:

$\lim\limits_{x \to -1/2} \frac{8x^3 + 1}{2x + 1}$

Answer

$= \frac{8\left(-\frac{1}{2}\right)^3 + 1}{2\left(-\frac{1}{2}\right) + 1}$

$= \frac{-1 + 1}{-1 + 1}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim\limits_{x \to -\frac{1}{2}} \frac{8x^3 + 1}{2x + 1}$

$= \lim\limits_{x \to -\frac{1}{2}} \frac{(2x)^3 + (1)^3}{2x + 1}$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$= \lim\limits_{x \to -\frac{1}{2}} \frac{(2x + 1)((2x)^2 + (1)^2 - 2x)}{2x + 1}$

$= \lim\limits_{x \to -\frac{1}{2}} (2x)^2 + (1)^2 - 2x$

$= (2(\frac{-1}{2}))^2 + (1)^2 - 2(-\frac{1}{2})$

= 1 + 1 + 1

= 3

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to -\frac{1}{2}} \frac{d(8x^3 + 1)}{d(2x + 1)}$

$= \lim_{x \to -\frac{1}{2}} \frac{24x^2}{2}$

$= \lim_{x \to -\frac{1}{2}} 12x^2$

= 12(-1/2)^2

= 12/4

= 3

6. Question

Evaluate the following limits:

$$\lim_{x \to 4} \frac{x^2 - 7x + 12}{x^2 - 3x - 4}$$

Answer

$= \frac{(4)^2 - 7(4) + 12}{(4)^2 - 3(4) - 4}$

$= \frac{28 - 28}{-16 + 16}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 4} \frac{(x^2 - 7x + 12)}{(x^2 - 3x - 4)}$

$= \lim_{x \to 4} \frac{(x^2 - 3x - 4x + 12)}{(x^2 - 4x + x - 4)}$

$= \lim_{x \to 4} \frac{x(x - 3) - 4(x - 3)}{x(x - 4) + 1(x - 4)}$

$= \lim_{x \to 4} \frac{(x - 3)(x - 4)}{(x - 4)(x + 1)}$

$= \lim_{x \to 4} \frac{(x - 3)}{(x + 1)}$

$= \frac{(4 - 3)}{(4 + 1)}$

$= \frac{1}{5}$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to 4} \frac{d(x^2 - 7x + 12)}{d(x^2 - 3x - 4)}$

$= \lim_{x \to 4} \frac{2x - 7}{2x - 3}$

$= \frac{2(4) - 7}{2(4) - 3}$

$= \frac{1}{5}$

7. Question

Evaluate the following limits:

$\lim_{x \to 2} \frac{x^4 - 16}{x - 2}$

Answer

$= \frac{(2)^4 - 16}{2 - 2}$

$= \frac{16 - 16}{2 - 2}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 2} \frac{(x^4 - 16)}{(x - 2)}$

$= \lim_{x \to 2} \frac{(x^4 - 2^4)}{(x - 2)}$

$= \lim_{x \to 2} \frac{(x^2)^2 - (2^2)^2}{(x - 2)}$

Since $a^2 - b^2 = (a + b)(a - b)$

$= \lim_{x \to 2} \frac{(x^2 - 2^2)(x^2 + 2^2)}{(x - 2)}$

$= \lim_{x \to 2} \frac{(x - 2)(x + 2)(x^2 + 2^2)}{(x - 2)}$

$= \lim_{x \to 2} (x + 2)(x^2 + 2^2)$

$= (2 + 2)(2^2 + 2^2)$

$= 32$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to 2} \frac{d(x^4-16)}{d(x-2)}$

$= \lim_{x \to 2} \frac{4x^3}{1}$

$= \lim_{x \to 2} 4x^3$

$= 4(2)^3$

= 32

8. Question

Evaluate the following limits:

$$\lim_{x \to 5} \frac{x^2 - 9x + 20}{x^2 - 6x + 5}$$

Answer

$= \frac{(5)^2-9(5)+20}{(5)^2-6(5)+5}$

$= \frac{45-45}{30-30}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 5} \frac{(x^2-9x+20)}{(x^2-6x+5)}$

$= \lim_{x \to 5} \frac{(x^2 - 5x - 4x + 20)}{(x^2 - 5x - x + 5)}$

$= \lim_{x \to 5} \frac{x(x-5) - 4(x-5)}{x(x-5) - 1(x-5)}$

$= \lim_{x \to 5} \frac{(x-5)(x-4)}{(x-5)(x-1)}$

$= \lim_{x \to 5} \frac{(x-4)}{(x-1)}$

$= \frac{(5-4)}{(5-1)}$

$= \frac{1}{4}$

Method 2:

By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to 5} \frac{d(x^2-9x+20)}{d(x^2-6x+5)}$

$= \lim_{x \to 5} \frac{2x-9}{2x-6}$

$= \frac{2(5)-9}{2(5)-6}$

$= \frac{1}{4}$

9. Question

Evaluate the following limits:

$$\lim_{x\to -1} \frac{x^3+1}{x+1}$$

Answer

$= \frac{(-1)^3+1}{-1+1}$

$= \frac{-1+1}{-1+1}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x\to -1} \frac{(x^3 + 1)}{(x + 1)}$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$= \lim_{x\to -1} \frac{(x + 1)(x^2 + 1^2 - x)}{(x + 1)}$

$= \lim_{x\to -1} (x^2 + 1^2 - x)$

$= (-1)^2 + (1)^2 - (-1)$

= 3

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x\to -1} \frac{d(x^3+1)}{d(x+1)}$

$= \lim_{x\to -1} \frac{3x^2}{1}$

$= \lim_{x\to -1} 3x^2$

$= 3(-1)^2$

= 3

10. Question

Evaluate the following limits:

$$\lim_{x\to 5} \frac{x^3-125}{x^2-7x+10}$$

Answer

$= \frac{(5)^3-125}{(5)^2-7(5)+10}$

$= \frac{125-125}{35-35}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$$= \lim_{x\to 5} \frac{(x^3-125)}{(x^2-7x+10)}$$

$$= \lim_{x\to 5} \frac{(x^3-5^3)}{(x^2-5x-2x+10)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x\to 5} \frac{(x-5)(x^2+5^2+5x)}{(x^2-5x-2x+10)}$$

$$= \lim_{x\to 5} \frac{(x-5)(x^2+5^2+5x)}{x(x-5)-2(x-5)}$$

$$= \lim_{x\to 5} \frac{(x-5)(x^2+5^2+5x)}{(x-5)(x-2)}$$

$$= \lim_{x\to 5} \frac{(x^2+5^2+5x)}{(x-2)}$$

$= \frac{(5^2+5^2+5(5))}{(5-2)}$

$= \frac{3.5^2}{3}$

= 25

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x\to 5} \frac{d(x^3-125)}{d(x^2-7x+10)}$$

$$= \lim_{x\to 5} \frac{3x^2}{2x-7}$$

$= \frac{3(5^2)}{2(5)-7}$

$= \frac{75}{3}$

= 25

11. Question

Evaluate the following limits:

$$\lim_{x\to\sqrt{2}} \frac{x^2-2}{x^2+\sqrt{2}x-4}$$

Answer

$= \frac{(\sqrt{2})^2 - 2}{(\sqrt{2})^2 + \sqrt{2}(\sqrt{2}) - 4}$

$= \frac{2-2}{4-4}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to \sqrt{2}} \frac{(x^2 - 2)}{(x^2 + \sqrt{2}x - 4)}$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$= \lim_{x \to \sqrt{2}} \frac{(x^2 - (\sqrt{2})^2)}{(x^2 + 2\sqrt{2}x - \sqrt{2}x - 4)}$

$= \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})}{x(x + 2\sqrt{2}) - \sqrt{2}(x + 2\sqrt{2})}$

$= \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})}{(x + 2\sqrt{2})(x - \sqrt{2})}$

$= \lim_{x \to \sqrt{2}} \frac{(x + \sqrt{2})}{(x + 2\sqrt{2})}$

$= \frac{(\sqrt{2} + \sqrt{2})}{(\sqrt{2} + 2\sqrt{2})}$

$= \frac{(2\sqrt{2})}{(3\sqrt{2})}$

$= \frac{(2)}{(3)}$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to \sqrt{2}} \frac{d(x^2 - 2)}{d(x^2 + \sqrt{2}x - 4)}$

$= \lim_{x \to \sqrt{2}} \frac{2x}{2x + \sqrt{2}}$

$= \frac{2(\sqrt{2})}{2(\sqrt{2}) + \sqrt{2}}$

$= \frac{2\sqrt{2}}{3\sqrt{2}}$

$= \frac{2}{3}$

12. Question

Evaluate the following limits:

$$\lim_{x\to\sqrt{3}} \frac{x^2-3}{x^2+3\sqrt{3}x-12}$$

Answer

$$= \frac{(\sqrt{3})^2-3}{(\sqrt{3})^2+3\sqrt{3}(\sqrt{3})-12}$$

$$= \frac{3-3}{12-12}$$

Since the form is

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x\to\sqrt{3}} \frac{(x^2-3)}{(x^2+3\sqrt{3}x-12)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x\to\sqrt{3}} \frac{(x^2-(\sqrt{3})^2)}{(x^2+4\sqrt{3}x-\sqrt{3}x-12)}$$

$$= \lim_{x\to\sqrt{3}} \frac{(x-\sqrt{3})(x+\sqrt{3})}{x(x+4\sqrt{3})-\sqrt{3}(x+4\sqrt{3})}$$

$$= \lim_{x\to\sqrt{3}} \frac{(x-\sqrt{3})(x+\sqrt{3})}{(x+4\sqrt{3})(x-\sqrt{3})}$$

$$= \lim_{x\to\sqrt{3}} \frac{(x+\sqrt{3})}{(x+4\sqrt{3})}$$

$$= \frac{(\sqrt{3}+\sqrt{3})}{(\sqrt{3}+4\sqrt{3})}$$

$$= \frac{(2\sqrt{3})}{(5\sqrt{3})}$$

$$= \frac{(2)}{(5)}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x\to\sqrt{3}} \frac{d(x^2-3)}{d(x^2+3\sqrt{3}x-12)}$$

$$= \lim_{x\to\sqrt{3}} \frac{2x}{2x+3\sqrt{3}}$$

$$= \frac{2(\sqrt{3})}{2(\sqrt{3})+3\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{5\sqrt{3}}$$

$$= \frac{2}{5}$$

13. Question

Evaluate the following limits:

$$\lim_{x \to \sqrt{3}} \frac{x^4 - 9}{x^2 + 4\sqrt{3}x - 15}$$

Answer

$$= \frac{(\sqrt{3})^4 - 9}{(\sqrt{3})^2 + 4\sqrt{3}(\sqrt{3}) - 15}$$

$$= \frac{9-9}{15-15}$$

Since the form is

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \to \sqrt{3}} \frac{(x^4 - 9)}{(x^2 + 4\sqrt{3}x - 15)}$$

$$= \lim_{x \to \sqrt{3}} \frac{(x^2)^2 - \left(\sqrt{3^2}\right)^2}{(x^2 + 4\sqrt{3}x - 15)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \to \sqrt{3}} \frac{(x^2 + (\sqrt{3})^2)(x^2 - (\sqrt{3})^2)}{(x^2 + 5\sqrt{3}x - \sqrt{3}x - 15)}$$

$$= \lim_{x \to \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})(x^2 + (\sqrt{3})^2)}{x(x + 5\sqrt{3}) - \sqrt{3}(x + 5\sqrt{3})}$$

$$= \lim_{x \to \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})(x^2 + (\sqrt{3})^2)}{(x + 5\sqrt{3})(x - \sqrt{3})}$$

$$= \lim_{x \to \sqrt{3}} \frac{(x + \sqrt{3})(x^2 + (\sqrt{3})^2)}{(x + 5\sqrt{3})}$$

$$= \frac{(\sqrt{3} + \sqrt{3})(\sqrt{3}^2 + (\sqrt{3})^2)}{(\sqrt{3} + 5\sqrt{3})}$$

$$= \frac{(2\sqrt{3})(2.3)}{(6\sqrt{3})}$$

= 2

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \to \sqrt{3}} \frac{d(x^4 - 9)}{d(x^2 + 4\sqrt{3}x - 15)}$$

$$= \lim_{x \to \sqrt{3}} \frac{4x^3}{2x + 4\sqrt{3}}$$

$= \frac{4(\sqrt{3})^3}{2(\sqrt{3}) + 4\sqrt{3}}$

$= \frac{12\sqrt{3}}{6\sqrt{3}}$

= 2

14. Question

Evaluate the following limits:

$$\lim_{x \to 2}\left(\frac{x}{x-2} - \frac{4}{x^2 - 2x}\right)$$

Answer

$$= \lim_{x \to 2}\left(\frac{x}{x-2} - \frac{4}{x^2 - 2x}\right)$$

$$= \lim_{x \to 2}\left(\frac{x}{x-2} - \frac{4}{x(x-2)}\right)$$

$$= \lim_{x \to 2}\left(\frac{x}{1} - \frac{4}{x}\right)\left(\frac{1}{x-2}\right)$$

$$= \lim_{x \to 2}\left(\frac{x^2 - 4}{x}\right)\left(\frac{1}{x-2}\right)$$

Since $a^2-b^2 = (a + b)(a-b)$

$$= \lim_{x \to 2}\left(\frac{x^2 - 2^2}{x}\right)\left(\frac{1}{x-2}\right)$$

$$= \lim_{x \to 2}\left(\frac{x-2}{x}\right)\left(\frac{x+2}{x-2}\right)$$

$$= \lim_{x \to 2}\left(\frac{x+2}{x}\right)$$

$$= \frac{4}{2}$$

= 2

15. Question

Evaluate the following limits:

$$\lim_{x \to 1}\left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1}\right)$$

Answer

$$\lim_{x \to 1}\left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1}\right) = \lim_{x \to 1}\left(\frac{1}{x^2 + 2x - x - 2} - \frac{x}{x^3 - 1}\right)$$

$$\Rightarrow \lim_{x \to 1}\left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1}\right) = \lim_{x \to 1}\left(\frac{1}{x(x+2) - 1(x+2)} - \frac{x}{x^3 - 1}\right)$$

$$\Rightarrow \lim_{x \to 1}\left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1}\right) = \lim_{x \to 1}\left(\frac{1}{(x+2)(x-1)} - \frac{x}{(x-1)(x^2 + x + 1)}\right)$$

$$\Rightarrow \lim_{x\to1}\left(\frac{1}{x^2+x-2}-\frac{x}{x^3-1}\right) = \lim_{x\to1}\frac{1}{x-1}\left(\frac{1}{x+2}-\frac{x}{x^2+x+1}\right)$$

$$\Rightarrow \lim_{x\to1}\left(\frac{1}{x^2+x-2}-\frac{x}{x^3-1}\right) = \lim_{x\to1}\frac{1}{x-1}\left(\frac{x^2+x+1-x(x+2)}{(x+2)(x^2+x+1)}\right)$$

$$\lim_{x\to1}\left(\frac{1}{x^2+x-2}+\frac{x}{x^3-1}\right) = \frac{-1}{(x+2)(x^2+x+1)}$$

Hence, $\lim_{x\to1}\left(\frac{1}{x^2+x-2}+\frac{x}{x^3-1}\right) = \frac{-1}{9}$

16. Question

Evaluate the following limits:

$$\lim_{x\to3}\left(\frac{1}{x-3}-\frac{2}{x^2-4x+3}\right)$$

Answer

$$= \lim_{x\to3}\left(\frac{1}{x-3}-\frac{2}{x^2-3x-x+3}\right)$$

$$= \lim_{x\to3}\left(\frac{1}{x-3}-\frac{2}{x(x-3)-1(x-3)}\right)$$

$$= \lim_{x\to3}\left(\frac{1}{x-3}-\frac{2}{(x-3)(x-1)}\right)$$

$$= \lim_{x\to3}\frac{1}{x-3}\left(1-\frac{2}{(x-1)}\right)$$

$$= \lim_{x\to3}\frac{1}{x-3}\left(\frac{x-1-2}{(x-1)}\right)$$

$$= \lim_{x\to3}\frac{1}{x-3}\left(\frac{x-3}{(x-1)}\right)$$

$$= \lim_{x\to3}\left(\frac{1}{(x-1)}\right)$$

$$= \frac{1}{(3-1)}$$

$$= \frac{1}{2}$$

17. Question

Evaluate the following limits:

$$\lim_{x\to2}\left(\frac{1}{x-2}-\frac{2}{x^2-2x}\right)$$

Answer

$$= \lim_{x\to2}\left(\frac{1}{x-2}-\frac{2}{x^2-2x}\right)$$

$$= \lim_{x\to2}\left(\frac{1}{x-2}-\frac{2}{x(x-2)}\right)$$

$= \lim_{x \to 2} \left(\frac{1}{1} - \frac{2}{x}\right)\left(\frac{1}{x-2}\right)$

$= \lim_{x \to 2} \left(\frac{x-2}{x}\right)\left(\frac{1}{x-2}\right)$

$= \lim_{x \to 2} \left(\frac{1}{x}\right)$

$= \frac{1}{2}$

18. Question

Evaluate the following limits:

$$\lim_{x \to 1/4} \frac{4x-1}{2\sqrt{x}-1}$$

Answer

$= \frac{4\left(\frac{1}{4}\right)-1}{2\left(\sqrt{\frac{1}{4}}\right)-1}$

$= \frac{1-1}{1-1}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 1/4} \frac{(4x-1)}{(2\sqrt{x}-1)}$

$= \lim_{x \to 1/4} \frac{(2\sqrt{x})^2-(1)^2}{(2\sqrt{x}-1)}$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a-b)$

$= \lim_{x \to 1/4} \frac{(2\sqrt{x}-1)(2\sqrt{x}+1)}{(2\sqrt{x}-1)}$

$= \lim_{x \to 1/4} (2\sqrt{x} + 1)$

$= \left(2\sqrt{\frac{1}{4}} + 1\right)$

$= \left(\frac{2}{2} + 1\right)$

= 2

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to 1/4} \frac{d(4x-1)}{d(2\sqrt{x}-1)}$

$= \lim_{x \to \frac{1}{4}} \frac{4}{2\left(\frac{1}{2}\right)x^{-\frac{1}{2}}}$

$= \frac{4}{\left(1/\sqrt{\frac{1}{4}}\right)}$

= 2

19. Question

Evaluate the following limits:

$$\lim_{x \to 4} \frac{x^2 - 16}{\sqrt{x} - 2}$$

Answer

$= \frac{4^2 - 16}{(\sqrt{4}) - 2}$

$= \frac{16 - 16}{2 - 2}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 4} \frac{(x^2 - 16)}{(\sqrt{x} - 2)}$

$= \lim_{x \to 4} \frac{(x)^2 - (4)^2}{(\sqrt{x} - 2)}$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$= \lim_{x \to 4} \frac{(x - 4)(x + 4)}{(\sqrt{x} - 2)}$

$= \lim_{x \to 4} \frac{((\sqrt{x})^2 - (2)^2)(x + 4)}{(\sqrt{x} - 2)}$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$= \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)(x + 4)}{(\sqrt{x} - 2)}$

$= \lim_{x \to 4} (\sqrt{x} + 2)(x + 4)$

$= (\sqrt{4} + 2)(4 + 4)$

= (2 + 2)(4 + 4)

= 32

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to 4} \frac{d(x^2 - 16)}{d(\sqrt{x} - 2)}$

$= \lim_{x \to 4} \frac{2x}{\left(\frac{1}{2}\right)x^{-\frac{1}{2}}}$

$= \lim_{x \to 4} 4x^{\frac{3}{2}}$

$= 4(4)^{3/2}$

= 32

20. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{(a+x)^2 - a^2}{x}$$

Answer

$= \frac{(a)^2 - a^2}{0}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 0} \frac{(a+x)^2 - a^2}{x}$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a-b)$

$= \lim_{x \to 0} \frac{(a + x + a)(a + x - a)}{x}$

$= \lim_{x \to 0} \frac{(2a + x)(x)}{x}$

$= \lim_{x \to 0} (2a + x)$

= 2a + 0

= 2a

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim_{x \to 0} \frac{d((a+x)^2 - a^2)}{d(x)}$

$= \lim_{x \to 0} \frac{2(a+x)}{1}$

= 2(a + 0)

= 2a

21. Question

Evaluate the following limits:

$$\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^3 - 2x^2} \right)$$

Answer

$= \lim_{x\to 2}(\frac{1}{x-2} - \frac{4}{x^3 - 2x^2})$

$= \lim_{x\to 2}(\frac{1}{x-2} - \frac{4}{x^2(x-2)})$

$= \lim_{x\to 2}(\frac{1}{1} - \frac{4}{x^2})(\frac{1}{x-2})$

$= \lim_{x\to 2}(\frac{x^2-4}{x^2})(\frac{1}{x-2})$

Since $a^3 + b^3 = (a + b)(a^2 + b^2\text{-}ab)$ & $a^2\text{-}b^2 = (a + b)(a\text{-}b)$

$= \lim_{x\to 2}(\frac{x+2}{x^2})(\frac{x-2}{x-2})$

$= \lim_{x\to 2}(\frac{x+2}{x^2})$

$= \frac{4}{4}$

= 1

22. Question

Evaluate the following limits:

$$\lim_{x\to 3}\left(\frac{1}{x-3} - \frac{3}{x^2-3x}\right)$$

Answer

$= \lim_{x\to 3}(\frac{1}{x-3} - \frac{3}{x^2 - 3x})$

$= \lim_{x\to 3}(\frac{1}{x-3} - \frac{3}{x(x-3)})$

$= \lim_{x\to 3}(\frac{1}{1} - \frac{3}{x})(\frac{1}{x-3})$

$= \lim_{x\to 3}(\frac{x-3}{x})(\frac{1}{x-3})$

$= \lim_{x\to 3}(\frac{1}{x})$

$= \frac{1}{3}$

23. Question

Evaluate the following limits:

$$\lim_{x\to 1}\left(\frac{1}{x-1} - \frac{2}{x^2-1}\right)$$

Answer

$$= \lim_{x\to1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x\to1}\left(\frac{1}{x-1}-\frac{2}{(x+1)(x-1)}\right)$$

$$= \lim_{x\to1}\left(\frac{1}{1}-\frac{2}{x+1}\right)\left(\frac{1}{x-1}\right)$$

$$= \lim_{x\to1}\left(\frac{x-1}{x+1}\right)\left(\frac{1}{x-1}\right)$$

$$= \lim_{x\to1}\left(\frac{1}{x+1}\right)$$

$$= \frac{1}{2}$$

24. Question

Evaluate the following limits:

$$\lim_{x\to3}(x^2-9)\left(\frac{1}{x+3}+\frac{1}{x-3}\right)$$

Answer

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x\to3}(x+3)(x-3)\left(\frac{1}{x+3}+\frac{1}{x-3}\right)$$

$$= \lim_{x\to3}\left(\frac{(x+3)(x-3)}{x+3}+\frac{(x+3)(x-3)}{x-3}\right)$$

$$= \lim_{x\to3}\left(\frac{(x-3)}{1}+\frac{(x+3)}{1}\right)$$

$$= \lim_{x\to3} 2x$$

$= 6$

25. Question

Evaluate the following limits:

$$\lim_{x\to1}\frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$$

Answer

$$= \frac{(1)^4-3(1)^3+2}{(1)^3-5(1)^2+3(1)+1}$$

$$= \frac{3-3}{5-5}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \to 1} \frac{(x)^4 - 3(x)^3 + 2}{(x)^3 - 5(x)^2 + 3(x) + 1}$$

$$= \lim_{x \to 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1}$$

$$= \lim_{x \to 1} \frac{x^4 - 2x^3 - x^3 + 2}{x^3 - x^2 - 3x^2 - x^2 + 3x + 1}$$

$$= \lim_{x \to 1} \frac{x^3(x-1) - 2(x^3 - 1)}{x^2(x-1) - 1(x^2 - 1) - 3x(x-1)}$$

Since $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a + b)(a-b)$

$$= \lim_{x \to 1} \frac{x^3(x-1) - 2(x-1)(x^2 + 1^2 + x)}{x^2(x-1) - 1(x-1)(x + 1) - 3x(x-1)}$$

$$= \lim_{x \to 1} \frac{(x-1)(x^3 - 2(x^2 + 1^2 + x))}{(x-1)(x^2 - 1(x + 1) - 3x)}$$

$$= \lim_{x \to 1} \frac{x^3 - 2(x^2 + 1^2 + x)}{x^2 - 1(x + 1) - 3x}$$

$$= \frac{1^3 - 2(1^2 + 1^2 + 1)}{1^2 - 1(1 + 1) - 3(1)}$$

$$= \frac{1 - 2(3)}{1 - 1(2) - 3(1)}$$

$$= \frac{-5}{-4}$$

$$= \frac{5}{4}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \to 1} \frac{d((x)^4 - 3(x)^3 + 2)}{d((x)^3 - 5(x)^2 + 3(x) + 1)}$$

$$= \lim_{x \to 1} \frac{4x^3 - 9x^2}{3x^2 - 10x + 3}$$

$$= \frac{4(1)^3 - 9(1)^2}{3(1)^2 - 10(1) + 3}$$

$$= \frac{-5}{-4}$$

$$= \frac{5}{4}$$

26. Question

Evaluate the following limits:

$$\lim_{x \to 2} \frac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6}$$

Answer

$$= \frac{(2)^3 + 3(2)^2 - 9(2) - 2}{(2)^3 - 2 - 6}$$

$= \frac{20-20}{8-8}$

Since the form is indeterminant

$= \frac{0}{0}$

Method 1: factorization

$= \lim\limits_{x\to2} \frac{(x)^3 + 3(x)^2 - 9(x) - 2}{(x)^3 - x - 6}$

By long division method

$= \lim\limits_{x\to2} 1 + \frac{3x^2 - 8x + 4}{x^3 - x - 6}$

$= \lim\limits_{x\to2} 1 + \frac{3x^2 - 6x - 2x + 4}{x^3 - 4x + 3x - 6}$

$= \lim\limits_{x\to2} 1 + \frac{3x(x-2) - 2(x-2)}{x(x^2-4) + 3(x-2)}$

Since $a^3-b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2-b^2 = (a + b)(a-b)$

$= \lim\limits_{x\to2} 1 + \frac{(x-2)(3x-2)}{x(x^2-2^2) + 3(x-2)}$

$= \lim\limits_{x\to2} 1 + \frac{(x-2)(3x-2)}{x(x-2)(x+2) + 3(x-2)}$

$= \lim\limits_{x\to2} 1 + \frac{(x-2)(3x-2)}{(x-2)[x(x+2) + 3]}$

$= \lim\limits_{x\to2} 1 + \frac{(3x-2)}{[x(x+2) + 3]}$

$= 1 + \frac{(3.2-2)}{[2(2+2)+3]}$

$= 1 + \frac{4}{11}$

$= \frac{15}{11}$

Method2: By L hospital rule:

Differentiating numerator and denominator separately:

$= \lim\limits_{x\to2} \frac{d((x)^3 + 3(x)^2 - 9(x) - 2)}{d((x)^3 - x - 6)}$

$= \lim\limits_{x\to2} \frac{3x^2 + 6x - 9}{3x^2 - 1}$

$= \frac{3(2)^2 + 6(2) - 9}{3(2)^2 - 1}$

$= \frac{24-9}{12-1}$

$= \frac{15}{11}$

27. Question

Evaluate the following limits:

$$\lim_{x\to 1} \frac{1-x^{-1/3}}{1-x^{-2/3}}$$

Answer

$$= \frac{-(1)^{-\frac{1}{3}}+1}{-(1)^{-\frac{2}{3}}+1}$$

$$= \frac{-1+1}{-1+1}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x\to 1} \frac{-(x)^{-\frac{1}{3}} + 1}{-\left((x)^{-\frac{1}{3}}\right)^{2} + 1}$$

Since $a^3-b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2-b^2 = (a + b)(a-b)$

$$= \lim_{x\to 1} \frac{-(x)^{-\frac{1}{3}} + 1}{\left[-(x)^{-\frac{1}{3}} + 1\right]\left[(x)^{-\frac{1}{3}} + 1\right]}$$

$$= \lim_{x\to 1} \frac{1}{\left[(x)^{-\frac{1}{3}} + 1\right]}$$

$$= \frac{1}{\left[(x)^{-\frac{1}{3}} + 1\right]}$$

$$= \frac{1}{[1 + 1]}$$

$$= \frac{1}{[2]}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x\to 1} \frac{d(-(x)^{-\frac{1}{3}}+1)}{d(-(x)^{-\frac{2}{3}}+1)}$$

$$= \lim_{x\to 1} \frac{\frac{1}{3}x^{-\frac{4}{3}}}{\frac{2}{3}x^{-\frac{5}{3}}}$$

$$= \lim_{x\to 1} \frac{1}{2} x^{\frac{1}{3}}$$

$$= \frac{1}{2}(1)^{\frac{1}{3}}$$

$$= \frac{1}{2}$$

28. Question

Evaluate the following limits:

$$\lim_{x\to3} \frac{x^2 - x - 6}{x^3 - 3x^2 + x - 3}$$

Answer

$$= \frac{(3)^2-(3)-6}{(3)^3-3(3)^2+3-3}$$

$$= \frac{9-9}{12-12}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x\to3}\frac{\{(x)^2-(x)-6\}}{\{(x)^3-3(x)^2+x-3\}}$$

$$= \lim_{x\to3}\frac{\{x^2-3x+2x-6\}}{\{x^3-3x^2+x-3\}}$$

$$= \lim_{x\to3}\frac{\{x(x-3)+2(x-3)\}}{\{x^2(x-3)+1(x-3)\}}$$

$$= \lim_{x\to3}\frac{\{(x+2)(x-3)\}}{\{(x^2+1)(x-3)\}}$$

$$= \lim_{x\to3}\frac{\{x+2\}}{\{x^2+1\}}$$

$$= \frac{\{3+2\}}{\{3^2+1\}}$$

$$= \frac{5}{10}$$

$$= \frac{1}{2}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x\to3}\frac{d\{(x)^2-(x)-6\}}{d\{(x)^3-3(x)^2+x-3]}$$

$$= \lim_{x\to3}\frac{2x-1}{3x^2-6x+1}$$

$$= \frac{2(3)-1}{3(3)^2-6(3)+1}$$

$$= \frac{5}{10}$$

$$= \frac{1}{2}$$

29. Question

Evaluate the following limits:

$$\lim_{x \to -2} \frac{x^3 + x^2 + 4x + 12}{x^3 - 3x + 2}$$

Answer

$$= \frac{(-2)^3 + (-2)^2 + 4(-2) + 12}{(-2)^3 - 3(-2) + 2}$$

$$= \frac{16-16}{8-8}$$

Since the form is

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \to -2} \frac{\{x^3 + x^2 + 4x + 12\}}{\{x^3 - 3x + 2\}}$$

By long division method

$$= \lim_{x \to -2} 1 + \frac{\{x^2 + 7x + 10\}}{\{x^3 - 3x + 2\}}$$

$$= \lim_{x \to -2} 1 + \frac{\{x^2 + 5x + 2x + 10\}}{\{x^3 - 4x + x + 2\}}$$

$$= \lim_{x \to -2} 1 + \frac{\{x(x + 5) + 2(x + 5)\}}{\{x(x^2 - 2^2) + 1(x + 2)\}}$$

Since $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a + b)(a-b)$

$$= \lim_{x \to -2} 1 + \frac{\{(x + 5)(x + 2)\}}{\{x(x + 2)(x - 2) + 1(x + 2)\}}$$

$$= \lim_{x \to -2} 1 + \frac{\{(x + 5)(x + 2)\}}{(x + 2)\{x(x - 2) + 1\}}$$

$$= \lim_{x \to -2} 1 + \frac{\{(x + 5)\}}{\{x(x - 2) + 1\}}$$

$$= 1 + \frac{\{(-2 + 5)\}}{\{-2(-2 - 2) + 1\}}$$

$$= 1 + \frac{3}{\{8 + 1\}}$$

$$= 1 + \frac{3}{\{9\}}$$

$$= 1 + \frac{1}{3}$$

$$= \frac{4}{3}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \to -2} \frac{d\{(x)^3 + (x)^2 + 4(x) + 12\}}{d\{(x)^3 - 3(x) + 2\}}$$

$= \lim_{x \to -2} \frac{3x^2 + 2x + 4}{3x^2 - 3}$

$= \frac{3(-2)^2 + 2(-2) + 4}{3(-2)^2 - 3}$

$= \frac{16 - 4}{12 - 3}$

$= \frac{12}{9} = \frac{4}{3}$

30. Question

Evaluate the following limits:

$$\lim_{x \to 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1}$$

Answer

$= \frac{(1)^3 + 3(1)^2 - 6(1) + 2}{(1)^3 + 3(1)^2 - 3(1) - 1}$

$= \frac{6 - 6}{3 - 3}$

Since the form is

$= \frac{0}{0}$

Method 1: factorization

$= \lim_{x \to 1} \frac{\{x^3 + 3x^2 - 6x + 2\}}{\{x^3 + 3x^2 - 3x - 1\}}$

by dividing

$= \lim_{x \to 1} 1 + \frac{-3x + 3}{\{x^3 - 1 + 3x^2 - 3x\}}$

Since $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a + b)(a-b)$

$= \lim_{x \to 1} 1 + \frac{-3x + 3}{(x - 1)(x^2 + 1 + x) + 3x(x - 1)}$

$= \lim_{x \to 1} 1 + \frac{-3(x - 1)}{(x - 1)[(x^2 + 1 + x) + 3x]}$

$= \lim_{x \to 1} 1 + \frac{-3}{[x^2 + 1 + 4x]}$

$= 1 + \frac{-3}{[1^2 + 1 + 4.1]}$

$= 1 + \frac{-3}{[6]}$

$= 1 + \frac{-1}{[2]}$

$= \frac{1}{2}$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x\to1} \frac{d\{(x)^3 + 3(x)^2 - 6(x) + 2\}}{d\{(x)^3 + 3(x)^2 - 3(x) - 1\}}$$

$$= \lim_{x\to1} \frac{3x^2 + 6x - 6}{3x^2 + 6x - 3}$$

$$= \frac{3(1)^2 + 6(1) - 6}{3(1)^2 + 6(1) - 3}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

31. Question

Evaluate the following limits:

$$\lim_{x\to2} \left\{\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x}\right\}$$

Answer

$$= \lim_{x\to2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x}\right)$$

$$= \lim_{x\to2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 2x^2 - x^2 + 2x}\right)$$

$$= \lim_{x\to2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^2(x-2) - x(x-2)}\right)$$

$$= \lim_{x\to2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{(x^2 - x)(x-2)}\right)$$

$$= \lim_{x\to2} \left(\frac{1}{1} - \frac{2(2x-3)}{(x^2 - x)}\right)\left(\frac{1}{x-2}\right)$$

$$= \lim_{x\to2} \left(\frac{x^2 - x - 4x + 6}{x^2 - x}\right)\left(\frac{1}{x-2}\right)$$

$$= \lim_{x\to2} \left(\frac{x^2 - 5x + 6}{x^2 - x}\right)\left(\frac{1}{x-2}\right)$$

$$= \lim_{x\to2} \left(\frac{x^2 - 2x - 3x + 6}{x^2 - x}\right)\left(\frac{1}{x-2}\right)$$

$$= \lim_{x\to2} \left(\frac{x(x-2) - 3(x-2)}{x^2 - x}\right)\left(\frac{1}{x-2}\right)$$

$$= \lim_{x\to2} \left(\frac{(x-3)(x-2)}{x^2 - x}\right)\left(\frac{1}{x-2}\right)$$

$$= \lim_{x\to2} \left(\frac{x-3}{x^2 - x}\right)$$

$$= \frac{2-3}{4-2}$$

$= \frac{-1}{2}$

32. Question

Evaluate the following limits:

$$\lim_{x \to 1} \frac{\sqrt{x^2-1}+\sqrt{x-1}}{\sqrt{x^2-1}}, x > 1$$

Answer

$$= \lim_{x \to 1} \frac{\sqrt{x^2 - 1} + \sqrt{x - 1}}{\sqrt{x^2 - 1}}$$

$$= \lim_{x \to 1} \frac{\sqrt{(x + 1)(x - 1)} + \sqrt{x - 1}}{\sqrt{(x - 1)(x + 1)}}$$

$$= \lim_{x \to 1} \frac{(\sqrt{(x + 1)} + 1)\sqrt{x - 1}}{\sqrt{(x - 1)(x + 1)}}$$

$$= \lim_{x \to 1} \frac{(\sqrt{(x + 1)} + 1)}{\sqrt{(x + 1)}}$$

$$= \frac{(\sqrt{(1 + 1)} + 1)}{\sqrt{(1 + 1)}}$$

$$= \frac{\sqrt{2} + 1}{\sqrt{2}}$$

33. Question

Evaluate the following limits:

$$\lim_{x \to 1} \left\{\frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x}\right\}$$

Answer

$$= \lim_{x \to 1} \left(\frac{x - 2}{x^2 - x} - \frac{1}{x^3 - 3x^2 + 2x}\right)$$

$$= \lim_{x \to 1} \left(\frac{x - 2}{x^2 - x} - \frac{1}{x^3 - 2x^2 - x^2 + 2x}\right)$$

$$= \lim_{x \to 1} \left(\frac{x - 2}{x^2 - x} - \frac{1}{x^2(x - 2) - x(x - 2)}\right)$$

$$= \lim_{x \to 1} \left(\frac{x - 2}{x^2 - x} - \frac{1}{(x^2 - x)(x - 2)}\right)$$

$$= \lim_{x \to 1} \left(\frac{x - 2}{1} - \frac{1}{(x - 2)}\right)\left(\frac{1}{x^2 - x}\right)$$

$$= \lim_{x \to 1} \left(\frac{(x - 2)^2 - 1}{x - 2}\right)\left(\frac{1}{x^2 - x}\right)$$

$$= \lim_{x\to 1} \left(\frac{1}{x^2 - x}\right)\left(\frac{x^2 - 4x + 3}{x - 2}\right)$$

$$= \lim_{x\to 1} \left(\frac{1}{x(x-1)}\right)\left(\frac{x^2 - 3x - x + 3}{x - 2}\right)$$

$$= \lim_{x\to 1} \left(\frac{1}{x(x-1)}\right)\left(\frac{x(x-3) - 1(x-3)}{x - 2}\right)$$

$$= \lim_{x\to 1} \left(\frac{1}{x(x-1)}\right)\left(\frac{(x-1)(x-3)}{x - 2}\right)$$

$$= \lim_{x\to 1} \left(\frac{x-3}{x(x-2)}\right)$$

$$= \frac{1-3}{1(1-2)}$$

= 2

34. Question

Evaluate the following limits:

$$\lim_{x\to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$$

Answer

$$= \frac{(1)^7 - 2(1)^5 + 1}{(1)^3 - 3(1)^2 + 2}$$

$$= \frac{2-2}{3-3}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x\to 1} \frac{\{(x)^7 - 2(x)^5 + 1\}}{\{(x)^3 - 3(x)^2 + 2\}}$$

$$= \lim_{x\to 1} \frac{\{(x)^7 - 1(x)^5 - x^5 + 1\}}{\{(x)^3 - (x)^2 - 2x^2 + 2\}}$$

$$= \lim_{x\to 1} \frac{\{(x)^5(x^2 - 1) - (x^5 - 1)\}}{\{(x)^3 - (x)^2 - 2x^2 + 2\}}$$

$$= \lim_{x\to 1} \frac{\{(x)^5(x^2 - 1) - (x-1)(x^4 + x^3 + x^2 + x + 1)\}}{\{(x)^3 - (x)^2 - 2x^2 + 2\}}$$

Since $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a + b)(a-b)$

$$= \lim_{x\to 1} \frac{\{(x)^5(x-1)(x+1) - (x-1)(x^4 + x^3 + x^2 + x + 1)\}}{x^2(x-1) - 2(x^2 - 1)}$$

$$= \lim_{x\to 1} \frac{\{(x)^5(x-1)(x+1) - (x-1)(x^4 + x^3 + x^2 + x + 1)\}}{x^2(x-1) - 2(x-1)(x+1)}$$

$$= \lim_{x \to 1} \frac{(x-1)\{(x)^5(x+1)-(x^4+x^3+x^2+x+1)\}}{(x-1)[x^2-2(x+1)]}$$

$$= \lim_{x \to 1} \frac{\{(x)^5(x+1)-(x^4+x^3+x^2+x+1)\}}{[x^2-2(x+1)]}$$

$$= \frac{\{(1)^5(1+1)-(1^4+1^3+1^2+1+1)\}}{[1^2-2(1+1)]}$$

$$= \frac{2-5}{1-4}$$

$$= \frac{-3}{-3}$$

= 1

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \to 1} \frac{d\{(x)^7-2(x)^5+1\}}{d\{(x)^3-3(x)^2+2\}}$$

$$= \lim_{x \to 1} \frac{7x^6-10x^4}{3x^2-6x}$$

$$= \frac{7(1)^6-10(1)^4}{3(1)^2-6(1)}$$

$$= \frac{-3}{-3}$$

= 1

Exercise 29.4

1. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sqrt{1+x+x^2}-1}{x}$$

Answer

Given $\lim_{x \to 0} \frac{\sqrt{1+x+x^2}-1}{x}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \to 0} \frac{\sqrt{1+x+x^2}-1}{x} = \lim_{x \to 0} \frac{(\sqrt{1+x+x^2}-1)}{x} \frac{(\sqrt{1+x+x^2}+1)}{(\sqrt{1+x+x^2}+1)}$$

Formula: (a + b) (a - b) = $a^2 - b^2$

$$= \lim_{x \to 0} \frac{1+x+x^2-1}{x(\sqrt{1+x+x^2}+1)}$$

$$= \lim_{x \to 0} \frac{x(1+x)}{x(\sqrt{1+x+x^2}+1)}$$

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{1+x+x^2}\ -\ 1}{x} = \lim_{x\to 0}\frac{(1+x)}{\left(\sqrt{1+x+x^2}+1\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get, $\lim_{x\to 0}\frac{\sqrt{1+x+x^2}\ -1}{x} = \frac{1}{1+1} = \frac{1}{2}$

2. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{2x}{\sqrt{a+x}-\sqrt{a-x}}$$

Answer

Given $\lim_{x\to 0}\frac{2x}{\sqrt{a+x}-\sqrt{a-x}}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to 0}\frac{2x}{\sqrt{a+x}-\sqrt{a-x}} = \lim_{x\to 0}\frac{2x}{(\sqrt{a+x}-\sqrt{a-x})}\frac{(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a+x}+\sqrt{a-x})}$$

Formula: (a + b) (a - b) = $a^2 - b^2$

$$= \lim_{x\to 0}\frac{2x(\sqrt{a+x}+\sqrt{a-x})}{a+x-a+x}$$

$$= \lim_{x\to 0}\frac{2x(\sqrt{a+x}+\sqrt{a-x})}{2x}$$

$$= \lim_{x\to 0}\frac{(\sqrt{a+x}+\sqrt{a-x})}{1}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim_{x\to 0}\frac{2x}{\sqrt{a+x}-\sqrt{a-x}} = \sqrt{a}+\sqrt{a} = 2\sqrt{a}$

3. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sqrt{a^2+x^2}-a}{x^2}$$

Answer

Given $\lim_{x\to 0}\frac{\sqrt{a^2+x^2}\ -a}{x^2}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{a^2+x^2}\ -\ a}{x^2} = \lim_{x\to 0}\frac{(\sqrt{a^2+x^2}\ -\ a)}{x^2}\frac{(\sqrt{a^2+x^2}\ +\ a)}{(\sqrt{a^2+x^2}\ +\ a)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 0}\frac{(a^2+x^2 - a^2)}{x^2(\sqrt{a^2+x^2} + a)}$$

$$= \lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{a^2+x^2} + a\right)}$$

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{a^2+x^2} - a}{x^2} = \lim_{x\to 0}\frac{1}{\left(\sqrt{a^2+x^2} + a\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get, $\lim_{x\to 0}\frac{\sqrt{a^2+x^2} - a}{x^2} = \frac{1}{a+a} = \frac{1}{2a}$

4. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$$

Answer

Given $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{(1-x)}}{2x}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0 we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{1+x} - \sqrt{(1-x)}}{2x} = \lim_{x\to 0}\frac{\left(\sqrt{1+x} - \sqrt{(1-x)}\right)}{2x}\frac{\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}{\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 0}\frac{1+x-1+x}{2x\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}$$

$$= \lim_{x\to 0}\frac{2x}{2x\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}$$

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{1+x} - \sqrt{(1-x)}}{2x} = \lim_{x\to 0}\frac{1}{\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{(1-x)}}{2x} = \frac{1}{1+1} = \frac{1}{2}$

5. Question

Evaluate the following limits:

$$\lim_{x\to 2}\frac{\sqrt{3-x}-1}{2-x}$$

Answer

Given $\lim_{x \to 2} \frac{\sqrt{3-x} - 1}{2-x}$

To find: the limit of the given equation when x tends to 2

Substituting x as 2, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\lim_{x \to 2} \frac{\sqrt{3-x} - 1}{2-x} = \lim_{x \to 2} \frac{(\sqrt{3-x} - 1)}{(2-x)} \frac{(\sqrt{3-x} + 1)}{(\sqrt{3-x} + 1)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x \to 2} \frac{(3-x-1)}{(2-x)(\sqrt{3-x} + 1)}$$

$$= \lim_{x \to 2} \frac{(2-x)}{(2-x)(\sqrt{3-x} + 1)}$$

$$\Rightarrow \lim_{x \to 2} \frac{\sqrt{3-x} - 1}{2-x} = \lim_{x \to 2} \frac{1}{(\sqrt{3-x} + 1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 2

We get $\lim_{x \to 2} \frac{\sqrt{3-x} - 1}{2-x} = \frac{1}{1+1} = \frac{1}{2}$

6. Question

Evaluate the following limits:

$$\lim_{x \to 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}}$$

Answer

Given $\lim_{x \to 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}}$

To find: the limit of the given equation when x tends to 3

Substituting x as 3, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \to 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}} = \lim_{x \to 3} \frac{(x-3)}{(\sqrt{x-2} - \sqrt{4-x})} \frac{(\sqrt{x-2} + \sqrt{4-x})}{(\sqrt{x-2} + \sqrt{4-x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x \to 3} \frac{(x-3)}{(x-2-4+x)} \frac{(\sqrt{x-2} + \sqrt{4-x})}{(1)}$$

$$= \lim_{x \to 3} \frac{(x-3)}{(2x-6)} \frac{(\sqrt{x-2} + \sqrt{4-x})}{(1)}$$

$$= \lim_{x \to 3} \frac{(x-3)}{2(x-3)} \frac{(\sqrt{x-2} + \sqrt{4-x})}{(1)}$$

$$\Rightarrow \lim_{x \to 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}} = \lim_{x \to 3} \frac{(1)}{2} \frac{(\sqrt{x-2} + \sqrt{4-x})}{(1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 3

We get $\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}=\frac{1+1}{2}=1$

7. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{x-1}{\sqrt{x^2+3}-2}$$

Answer

Given $\lim_{x\to 0}\frac{x-1}{\sqrt{x^2+3}-2}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we find that it is in non-indeterminant form so by substituting x as 0 we will directly get the answer

$$\Rightarrow \lim_{x\to 0}\frac{x-1}{\sqrt{x^2+3}-2}=\frac{0-1}{\sqrt{0+3}-2}$$

We get $\lim_{x\to 0}\frac{x-1}{\sqrt{x^2+3}-2}=\frac{-1}{\sqrt{3}-2}$ as the answer

8. Question

Evaluate the following limits:

$$\lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x-1}$$

Answer

Given $\lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x-1}$

To find: the limit of the given equation when x tends to 1

Substituting x as 1, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x-1}=\lim_{x\to 1}\frac{(\sqrt{5x-4}-\sqrt{x})}{(x-1)}\frac{(\sqrt{5x-4}+\sqrt{x})}{(\sqrt{5x-4}+\sqrt{x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$=\lim_{x\to 1}\frac{(5x-4-x)}{(x-1)}\frac{1}{(\sqrt{5x-4}+\sqrt{x})}$$

$$=\lim_{x\to 1}\frac{4(x-1)}{(x-1)}\frac{1}{(\sqrt{5x-4}+\sqrt{x})}$$

$$\Rightarrow \lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x-1}=\lim_{x\to 1}\frac{4}{1}\frac{1}{(\sqrt{5x-4}+\sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x-1}=\frac{4}{1+1}=2$

9. Question

Evaluate the following limits:

$$\lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}-2}$$

Answer

Given $\lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}-2}$

To find: the limit of the given equation when x tends to 1

Substituting x as 1, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}\;-2} = \lim_{x\to1}\frac{(x-1)}{(\sqrt{x^2+3}\;-2)}\frac{(\sqrt{x^2+3}+2)}{(\sqrt{x^2+3}+2)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to1}\frac{(x-1)}{(x^2+3-4)}\frac{(\sqrt{x^2+3}+2)}{1}$$

$$= \lim_{x\to1}\frac{(x-1)}{(x-1)(x+1)}\frac{(\sqrt{x^2+3}+2)}{1}$$

$$\Rightarrow \lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}\;-2} = \lim_{x\to1}\frac{1}{(x+1)}\frac{(\sqrt{x^2+3}+2)}{1}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}-2} = \frac{4}{1+1} = 2$

10. Question

Evaluate the following limits:

$$\lim_{x\to3}\frac{\sqrt{x+3}-\sqrt{6}}{x^2-9}$$

Answer

Given $\lim_{x\to3}\frac{\sqrt{x+3}-\sqrt{6}}{x^2-9}$

To find: the limit of the given equation when x tends to 3

Substituting x as 3, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to3}\frac{\sqrt{x+3}-\sqrt{6}}{x^2-9} = \lim_{x\to3}\frac{(\sqrt{x+3}-\sqrt{6})}{(x^2-9)}\frac{(\sqrt{x+3}+\sqrt{6})}{(\sqrt{x+3}+\sqrt{6})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to3}\frac{(x+3-6)}{(x^2-9)}\frac{1}{(\sqrt{x+3}+\sqrt{6})}$$

$$= \lim_{x\to3}\frac{(x-3)}{(x-3)(x+3)}\frac{1}{(\sqrt{x+3}+\sqrt{6})}$$

$$\Rightarrow \lim_{x\to 3}\frac{\sqrt{x+3}-\sqrt{6}}{x^2-9}=\lim_{x\to 3}\frac{1}{(x+3)}\frac{1}{\left(\sqrt{x+3}+\sqrt{6}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 3

We get $\lim_{x\to 3}\frac{\sqrt{x+3}-\sqrt{6}}{x^2-9}=\frac{1}{6(2\sqrt{6})}=\frac{1}{12\sqrt{6}}$

11. Question

Evaluate the following limits:

$$\lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x^2-1}$$

Answer

Given $\lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x^2-1}$

To find: the limit of the given equation when x tends to 1

Substituting x as 1 we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x^2-1}=\lim_{x\to 1}\frac{\left(\sqrt{5x-4}-\sqrt{x}\right)}{(x^2-1)}\frac{\left(\sqrt{5x-4}+\sqrt{x}\right)}{\left(\sqrt{5x-4}+\sqrt{x}\right)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$=\lim_{x\to 1}\frac{(5x-4-x)}{(x^2-1)}\frac{1}{\left(\sqrt{5x-4}+\sqrt{x}\right)}$$

$$=\lim_{x\to 1}\frac{4(x-1)}{(x-1)(x+1)}\frac{1}{\left(\sqrt{5x-4}+\sqrt{x}\right)}$$

$$\Rightarrow \lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x^2-1}=\lim_{x\to 1}\frac{4}{(x+1)}\frac{1}{\left(\sqrt{5x-4}+\sqrt{x}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x\to 1}\frac{\sqrt{5x-4}-\sqrt{x}}{x^2-1}=\frac{4}{2(2)}=1$

12. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$$

Answer

Given $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{(\sqrt{1+x}-1)}{x}\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 0}\frac{(1+x-1)}{x}\frac{1}{(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0}\frac{x}{x}\frac{1}{(\sqrt{1+x}+1)}$$

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{1}{(\sqrt{1+x}+1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \frac{1}{1+1} = \frac{1}{2}$

13. Question

Evaluate the following limits:

$$\lim_{x\to 2}\frac{\sqrt{x^2+1}-\sqrt{5}}{x-2}$$

Answer

Given $\lim_{x\to 2}\frac{\sqrt{x^2+1}-\sqrt{5}}{x-2}$

To find: the limit of the given equation when x tends to 2

Substituting x as 2, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to 2}\frac{\sqrt{x^2+1}-\sqrt{5}}{x-2} = \lim_{x\to 2}\frac{(\sqrt{x^2+1}-\sqrt{5})}{x-2}\frac{(\sqrt{x^2+1}+\sqrt{5})}{(\sqrt{x^2+1}+\sqrt{5})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 2}\frac{(x^2+1-5)}{x-2}\frac{1}{(\sqrt{x^2+1}+\sqrt{5})}$$

$$= \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}\frac{1}{(\sqrt{x^2+1}+\sqrt{5})}$$

$$= \lim_{x\to 2}\frac{(x+2)}{1}\frac{1}{(\sqrt{x^2+1}+\sqrt{5})}$$

Now we can see that the indeterminant form is removed, so substituting x as 2

We get $\lim_{x\to 2}\frac{\sqrt{x^2+1}-\sqrt{5}}{x-2} = \frac{2+2}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$

14. Question

Evaluate the following limits:

$$\lim_{x\to 2}\frac{x-2}{\sqrt{x}-\sqrt{2}}$$

Answer

G iven $\lim\limits_{x\to2}\frac{x-2}{\sqrt{x}-\sqrt{2}}$

To find: the limit of the given equation when x tends to 2

Substituting x as 2, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to2}\frac{x-2}{\sqrt{x}-\sqrt{2}} = \lim_{x\to2}\frac{(x-2)}{(\sqrt{x}-\sqrt{2})}\frac{(\sqrt{x}+\sqrt{2})}{(\sqrt{x}+\sqrt{2})}$$

Formula: (a+ b) (a - b) = a^2 - b^2

$$= \lim_{x\to2}\frac{(x-2)}{(x-2)}\frac{(\sqrt{x}+\sqrt{2})}{1}$$

$$= \lim_{x\to2}\frac{(\sqrt{x}+\sqrt{2})}{1}$$

Now we can see that the indeterminant form is removed, so substituting x as 2

We get $\lim\limits_{x\to2}\frac{x-2}{\sqrt{x}-\sqrt{2}} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

15. Question

Evaluate the following limits:

$$\lim_{x\to7}\frac{4-\sqrt{9+x}}{1-\sqrt{8-x}}$$

Answer

Given $\lim\limits_{x\to7}\frac{4-\sqrt{9+x}}{1-\sqrt{8-x}}$

To find: the limit of the given equation when x tends to 7

Substituting x as 7, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x\to7}\frac{\left(4-\sqrt{9+x}\right)}{\left(1-\sqrt{8-x}\right)}\frac{(1+\sqrt{8-x})}{(1+\sqrt{8-x})}\frac{\left(4+\sqrt{9+x}\right)}{\left(4+\sqrt{9+x}\right)}$$

Formula: (a+b) (a-b) = a^2-b^2

$$= \lim_{x\to7}\frac{(16-9-x)}{(1-8+x)}\frac{(1+\sqrt{8-x})}{\left(4+\sqrt{9+x}\right)}$$

$$= \lim_{x\to7}\frac{(7-x)}{(-7+x)}\frac{(1+\sqrt{8-x})}{\left(4+\sqrt{9+x}\right)}$$

$$\Rightarrow \lim_{x\to7}\frac{4-\sqrt{9+x}}{1-\sqrt{8-x}} = \lim_{x\to7}\frac{-(1+\sqrt{8-x})}{\left(4+\sqrt{9+x}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 7

We get $\lim\limits_{x\to7}\frac{4-\sqrt{9+x}}{1-\sqrt{8-x}} = \frac{-2}{8} = -\frac{1}{4}$

16. Question

Evaluate the following limits:

$$\lim_{x\to 0} \frac{\sqrt{a+x}-\sqrt{a}}{x\sqrt{a^2+ax}}$$

Answer

Given $\lim_{x\to 0} \frac{\sqrt{a+x}-\sqrt{a}}{x\sqrt{a^2+ax}}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation,

$$\Rightarrow \lim_{x\to 0} \frac{\sqrt{a+x}-\sqrt{a}}{x\sqrt{a^2+ax}} = \lim_{x\to 0} \frac{(\sqrt{a+x}-\sqrt{a})}{(x\sqrt{a^2+ax})} \frac{(\sqrt{a+x}+\sqrt{a})}{(\sqrt{a+x}+\sqrt{a})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x\to 0} \frac{(a+x-a)}{(x\sqrt{a^2+ax})} \frac{1}{(\sqrt{a+x}+\sqrt{a})}$$

$$= \lim_{x\to 0} \frac{(x)}{(x\sqrt{a^2+ax})} \frac{1}{(\sqrt{a+x}+\sqrt{a})}$$

$$= \lim_{x\to 0} \frac{1}{(\sqrt{a^2+ax})(\sqrt{a+x}+\sqrt{a})}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim_{x\to 0} \frac{\sqrt{a+x}-\sqrt{a}}{x\sqrt{a^2+ax}} = \frac{1}{a(2\sqrt{a})} = \frac{1}{2a\sqrt{a}}$

17. Question

Evaluate the following limits:

$$\lim_{x\to 5} \frac{x-5}{\sqrt{6x-5}-\sqrt{4x+5}}$$

Answer

Given $\lim_{x\to 5} \frac{x-5}{\sqrt{6x-5}-\sqrt{4x+5}}$

To find: the limit of the given equation when x tends to 5

Substituting x as 5, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x\to 5} \frac{(x-5)}{(\sqrt{6x-5}-\sqrt{4x+5})} \frac{(\sqrt{6x-5}+\sqrt{4x+5})}{(\sqrt{6x-5}+\sqrt{4x+5})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x\to 5} \frac{(x-5)}{(6x-5-4x-5)} \frac{(\sqrt{6x-5}+\sqrt{4x+5})}{1}$$

$$= \lim_{x \to 5} \frac{(x-5)}{2(x-5)} \frac{(\sqrt{6x-5} + \sqrt{4x+5})}{1}$$

$$= \lim_{x \to 5} \frac{1}{2} \frac{(\sqrt{6x-5} + \sqrt{4x+5})}{1}$$

Now we can see that the indeterminant form is removed, so substituting x as 5

We get $\lim_{x \to 5} \frac{x-5}{\sqrt{6x-5} - \sqrt{4x+5}} = \frac{\sqrt{25}+\sqrt{25}}{2} = \frac{10}{2} = 5$

18. Question

Evaluate the following limits:

$$\lim_{x \to 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^3 - 1}$$

Answer

Given $\lim_{x \to 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x^3-1}$

To find: the limit of the given equation when x tends to 1

Substituting x as 1, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \to 1} \frac{(\sqrt{5x-4} - \sqrt{x})}{(x^3-1)} \frac{(\sqrt{5x-4} + \sqrt{x})}{(\sqrt{5x-4} + \sqrt{x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x \to 1} \frac{(5x-4-x)}{(x^3-1)} \frac{1}{(\sqrt{5x-4} + \sqrt{x})}$$

$$= \lim_{x \to 1} \frac{4(x-1)}{(x-1)(x^2+x+1)} \frac{1}{(\sqrt{5x-4} + \sqrt{x})}$$

$$= \lim_{x \to 1} \frac{4}{(x^2+x+1)} \frac{1}{(\sqrt{5x-4} + \sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x \to 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x^3-1} = \frac{4}{(1+1+1)(1+1)} = \frac{2}{3}$

19. Question

Evaluate the following limits:

$$\lim_{x \to 2} \frac{\sqrt{1+4x} - \sqrt{5+2x}}{x-2}$$

Answer

Given $\lim_{x \to 2} \frac{\sqrt{1+4x}-\sqrt{5+2x}}{x-2}$

To find: the limit of the given equation when x tends to 2

Substituting x as 2, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x\to 2} \frac{\left(\sqrt{1+4x} - \sqrt{5+2x}\right)}{(x-2)} \frac{\left(\sqrt{1+4x} + \sqrt{5+2x}\right)}{\left(\sqrt{1+4x} + \sqrt{5+2x}\right)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 2} \frac{(1+4x-5-2x)}{(x-2)} \frac{1}{\left(\sqrt{1+4x} + \sqrt{5+2x}\right)}$$

$$= \lim_{x\to 2} \frac{2(x-2)}{(x-2)} \frac{1}{\left(\sqrt{1+4x} + \sqrt{5+2x}\right)}$$

$$= \lim_{x\to 2} \frac{2}{1} \frac{1}{\left(\sqrt{1+4x} + \sqrt{5+2x}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 2

We get $\lim_{x\to 2} \frac{\sqrt{1+4x}-\sqrt{5+2x}}{x-2} = \frac{2}{(3+3)} = \frac{1}{3}$

20. Question

Evaluate the following limits:

$$\lim_{x\to 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1}$$

Answer

Given $\lim_{x\to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$

To find: the limit of the given equation when x tends to 1

Substituting x as 1, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1} = \lim_{x\to 1} \frac{(\sqrt{3+x} - \sqrt{5-x})}{(x^2-1)} \frac{(\sqrt{3+x} + \sqrt{5-x})}{(\sqrt{3+x} + \sqrt{5-x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 1} \frac{(3+x-5+x)}{(x^2-1)} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x\to 1} \frac{2(x-1)}{(x-1)(x+1)} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x\to 1} \frac{2}{(x+1)} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x\to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1} = \frac{2}{(2)(2+2)} = \frac{1}{4}$

21. Question

Evaluate the following limits:

$$\lim_{x\to 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x}$$

Answer

Given $\lim_{x \to 0} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation,

$$= \lim_{x \to 0} \frac{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)}{x} \frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x \to 0} \frac{(1+x^2-1+x^2)}{x} \frac{1}{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$$

$$\Rightarrow = \lim_{x \to 0} \frac{(2x^2)}{x} \frac{1}{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$$

$$= \lim_{x \to 0} \frac{(2x)}{1} \frac{1}{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim_{x \to 0} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x} = \frac{0}{2} = 0$

22. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sqrt{1+x+x^2}-\sqrt{x+1}}{2x^2}$$

Answer

Given $\lim_{x \to 0} \frac{\sqrt{1+x+x^2}-\sqrt{x+1}}{2x^2}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \to 0} \frac{\left(\sqrt{1+x+x^2}-\sqrt{x+1}\right)}{2x^2} \frac{\left(\sqrt{1+x+x^2}+\sqrt{x+1}\right)}{\left(\sqrt{1+x+x^2}+\sqrt{x+1}\right)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x \to 0} \frac{(1+x+x^2-x-1)}{2x^2} \frac{1}{\left(\sqrt{1+x+x^2}+\sqrt{x+1}\right)}$$

$$= \lim_{x \to 0} \frac{(x^2)}{2x^2} \frac{1}{\left(\sqrt{1+x+x^2}+\sqrt{x+1}\right)}$$

$$= \lim_{x \to 0} \frac{(1)}{2} \frac{1}{\left(\sqrt{1+x+x^2}+\sqrt{x+1}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim\limits_{x\to 0}\frac{\sqrt{1+x+x^2}-\sqrt{x+1}}{2x^2}=\frac{1}{2(1+1)}=\frac{1}{4}$

23. Question

Evaluate the following limits:

$$\lim_{x\to 4}\frac{2-\sqrt{x}}{4-x}$$

Answer

Given $\lim\limits_{x\to 4}\frac{2-\sqrt{x}}{4-x}$

To find: the limit of the given equation when x tends to 4

Substituting x as 4, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$=\lim_{x\to 4}\frac{(2-\sqrt{x})}{4-x}\frac{(2+\sqrt{x})}{(2+\sqrt{x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$=\lim_{x\to 4}\frac{(4-x)}{4-x}\frac{(1)}{(2+\sqrt{x})}$$

$$=\lim_{x\to 4}\frac{1}{1}\frac{(1)}{(2+\sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 4

We get $\lim\limits_{x\to 4}\frac{2-\sqrt{x}}{4-x}=\frac{1}{2(\sqrt{4})}=\frac{1}{4}$

24. Question

Evaluate the following limits:

$$\lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}$$

Answer

Given $\lim\limits_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}$

To find: the limit of the given equation when x tends to a

Substituting x as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}=\lim_{x\to a}\frac{(x-a)}{(\sqrt{x}-\sqrt{a})}\frac{(\sqrt{x}+\sqrt{a})}{(\sqrt{x}+\sqrt{a})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$=\lim_{x\to a}\frac{(x-a)}{(x-a)}\frac{(\sqrt{x}+\sqrt{a})}{(1)}$$

$$= \lim_{x \to a} \frac{1}{1} \frac{(\sqrt{x} + \sqrt{a})}{(1)}$$

Now we can see that the indeterminant form is removed, so substituting x as a

We get $\lim_{x \to a} \frac{x-a}{\sqrt{x}-\sqrt{a}} = \sqrt{a} + \sqrt{a} = 2\sqrt{a}$

25. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x}$$

Answer

Given $\lim_{x \to 0} \frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$

To find: the limit of the given equation when x tends to 0

Substituting 0 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \to 0} \frac{(\sqrt{1+3x} - \sqrt{1-3x})}{x} \frac{(\sqrt{1+3x} + \sqrt{1-3x})}{(\sqrt{1+3x} + \sqrt{1-3x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x \to 0} \frac{(1 + 3x - 1 + 3x)}{x} \frac{1}{(\sqrt{1+3x} + \sqrt{1-3x})}$$

$$= \lim_{x \to 0} \frac{(6x)}{x} \frac{1}{(\sqrt{1+3x} + \sqrt{1-3x})}$$

$$= \lim_{x \to 0} \frac{(6)}{1} \frac{1}{(\sqrt{1+3x} + \sqrt{1-3x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim_{x \to 0} \frac{\sqrt{1+3x}-\sqrt{1-3x}}{x} = \frac{6}{1+1} = 3$

26. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sqrt{2-x} - \sqrt{2+x}}{x}$$

Answer

Given $\lim_{x \to 0} \frac{\sqrt{2-x}-\sqrt{2+x}}{x}$

To find: the limit of the given equation when x tends to 0

Substituting 0 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \to 0} \frac{(\sqrt{2-x} - \sqrt{2+x})}{x} \frac{(\sqrt{2-x} + \sqrt{2+x})}{(\sqrt{2-x} + \sqrt{2+x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 0} \frac{(2-x-2-x)}{x} \frac{1}{(\sqrt{2-x}+\sqrt{2+x})}$$

$$= \lim_{x\to 0} \frac{(-2x)}{x} \frac{1}{(\sqrt{2-x}+\sqrt{2+x})}$$

$$= \lim_{x\to 0} \frac{(-2)}{1} \frac{1}{(\sqrt{2-x}+\sqrt{2+x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim_{x\to 0} \frac{\sqrt{2-x}-\sqrt{2+x}}{x} = \frac{-2}{\sqrt{2}+\sqrt{2}} = -\frac{1}{\sqrt{2}}$

27. Question

Evaluate the following limits:

$$\lim_{x\to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$$

Answer

Given $\lim_{x\to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$

To find: the limit of the given equation when x tends to 1

Substituting 1 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x\to 1} \frac{(\sqrt{3+x}-\sqrt{5-x})}{x^2-1} \frac{(\sqrt{3+x}+\sqrt{5-x})}{(\sqrt{3+x}+\sqrt{5-x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 1} \frac{(3+x-5+x)}{x^2-1} \frac{1}{(\sqrt{3+x}+\sqrt{5-x})}$$

$$= \lim_{x\to 1} \frac{2(x-1)}{(x-1)(x+1)} \frac{1}{(\sqrt{3+x}+\sqrt{5-x})}$$

$$= \lim_{x\to 1} \frac{2}{(x+1)} \frac{1}{(\sqrt{3+x}+\sqrt{5-x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x\to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1} = \frac{2}{2(2+2)} = \frac{1}{4}$

28. Question

Evaluate the following limits:

$$\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$$

Answer

Given $\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$

To find: the limit of the given equation when x tends to 1

Substituting 1 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x\to1} \frac{(2x-3)(\sqrt{x}-1)}{(3x^2+3x-6)} \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to1} \frac{(2x-3)(x-1)}{(3x^2+3x-6)} \frac{1}{(\sqrt{x}+1)}$$

$$= \lim_{x\to1} \frac{(2x-3)(x-1)}{3(x^2+x-2)} \frac{1}{(\sqrt{x}+1)}$$

$$= \lim_{x\to1} \frac{(2x-3)(x-1)}{3(x-1)(x+2)} \frac{1}{(\sqrt{x}+1)}$$

$$= \lim_{x\to1} \frac{(2x-3)}{3(x+2)} \frac{1}{(\sqrt{x}+1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x\to1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6} = \frac{2-3}{(3)(3)(2)} = \frac{-1}{18}$

29. Question

Evaluate the following limits:

$$\lim_{x\to0} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$$

Answer

Given $\lim_{x\to0} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$

To find: the limit of the given equation when x tends to 0

Substituting 0 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x\to0} \frac{(\sqrt{1+x^2}-\sqrt{1+x})}{(\sqrt{1+x^3}-\sqrt{1+x})} \frac{(\sqrt{1+x^2}+\sqrt{1+x})}{(\sqrt{1+x^2}+\sqrt{1+x})} \frac{(\sqrt{1+x^3}+\sqrt{1+x})}{(\sqrt{1+x^3}+\sqrt{1+x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to0} \frac{(1+x^2-1-x)}{(1+x^3-1-x)} \frac{(1)}{(\sqrt{1+x^2}+\sqrt{1+x})} \frac{(\sqrt{1+x^3}+\sqrt{1+x})}{(1)}$$

$$= \lim_{x\to0} \frac{(x^2-x)}{(x^3-x)} \frac{(1)}{(\sqrt{1+x^2}+\sqrt{1+x})} \frac{(\sqrt{1+x^3}+\sqrt{1+x})}{(1)}$$

$$= \lim_{x\to0} \frac{x(x-1)}{x(x^2-1)} \frac{(1)}{(\sqrt{1+x^2}+\sqrt{1+x})} \frac{(\sqrt{1+x^3}+\sqrt{1+x})}{(1)}$$

$$= \lim_{x\to 0}\frac{(x-1)}{(x^2-1)}\frac{(1)}{(\sqrt{1+x^2}+\sqrt{1+x})}\frac{(\sqrt{1+x^3}+\sqrt{1+x})}{(1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

We get $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}} = \frac{1+1}{1+1} = \frac{2}{2} = 1$

30. Question

Evaluate the following limits:

$$\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$$

Answer

Given $\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$

To find: the limit of the given equation when x tends to 1

Substituting 1 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x\to 1}\frac{(x^2-\sqrt{x})}{(\sqrt{x}-1)}\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}\frac{(x^2+\sqrt{x})}{(x^2+\sqrt{x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to 1}\frac{(x^4-x)}{(x-1)}\frac{(\sqrt{x}+1)}{(1)}\frac{(1)}{(x^2+\sqrt{x})}$$

$$= \lim_{x\to 1}\frac{x(x^3-1)}{(x-1)}\frac{(\sqrt{x}+1)}{(1)}\frac{(1)}{(x^2+\sqrt{x})}$$

$$= \lim_{x\to 1}\frac{x(x-1)(x^2+x+1)}{(x-1)}\frac{(\sqrt{x}+1)}{(1)}\frac{(1)}{(x^2+\sqrt{x})}$$

$$= \lim_{x\to 1}\frac{x(x^2+x+1)}{1}\frac{(\sqrt{x}+1)}{(1)}\frac{(1)}{(x^2+\sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1} = \frac{(3)(2)}{2} = 3$

31. Question

Evaluate the following limits:

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}, x \neq 0$$

Answer

Given $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

To find: the limit of the given equation when h tends to 0

Substituting 0 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h} \frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{h\to 0} \frac{(x+h-x)}{h} \frac{(1)}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{(h)}{h} \frac{(1)}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{(1)}{1} \frac{(1)}{(\sqrt{x+h}+\sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting h as 0

We get $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

32. Question

Evaluate the following limits:

$$\lim_{x\to\sqrt{10}} \frac{\sqrt{7+2x}-(\sqrt{5}+\sqrt{2})}{x^2-10}$$

Answer

Given $\lim_{x\to\sqrt{10}} \frac{\sqrt{7+2x}-(\sqrt{5}+\sqrt{2})}{x^2-10}$

To find: the limit of the given equation when x tends to √10

Re-writing the equation as

$$= \lim_{x\to\sqrt{10}} \frac{\sqrt{7+2x}-\sqrt{(\sqrt{5}+\sqrt{2})^2}}{x^2-10}$$

$$= \lim_{x\to\sqrt{10}} \frac{\sqrt{7+2x}-\sqrt{5+2+2\sqrt{10}}}{x^2-10}$$

$$= \lim_{x\to\sqrt{10}} \frac{\sqrt{7+2x}-\sqrt{7+2\sqrt{10}}}{x^2-10}$$

Now rationalizing the above equation

$$= \lim_{x\to\sqrt{10}} \frac{\left(\sqrt{7+2x}-\sqrt{7+2\sqrt{10}}\right)}{x^2-10} \frac{\left(\sqrt{7+2x}+\sqrt{7+2\sqrt{10}}\right)}{\left(\sqrt{7+2x}+\sqrt{7+2\sqrt{10}}\right)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to\sqrt{10}} \frac{\left(7+2x-\left(7+2\sqrt{10}\right)\right)}{x^2-10} \frac{(1)}{\left(\sqrt{7+2x}+\sqrt{7+2\sqrt{10}}\right)}$$

$$= \lim_{x\to\sqrt{10}} \frac{\left(2x-2\sqrt{10}\right)}{x^2-10} \frac{(1)}{\left(\sqrt{7+2x}+\sqrt{7+2\sqrt{10}}\right)}$$

$$= \lim_{x\to\sqrt{10}} \frac{2(x-\sqrt{10})}{(x+\sqrt{10})(x-\sqrt{10})} \frac{(1)}{\left(\sqrt{7+2x}+\sqrt{7+2\sqrt{10}}\right)}$$

$$= \lim_{x\to\sqrt{10}} \frac{2(1)}{(x+\sqrt{10})(1)} \frac{(1)}{\left(\sqrt{7+2x}+\sqrt{7+2\sqrt{10}}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as $\sqrt{10}$

$$= \frac{2}{2\sqrt{10}} \frac{1}{\left(2\sqrt{7+2\sqrt{10}}\right)}$$

$$= \frac{1}{2\sqrt{10}} \frac{1}{\left(\sqrt{7+2\sqrt{10}}\right)}$$

33. Question

Evaluate the following limits:

$$\lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6}$$

Answer

Given $\lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6}$

To find: the limit of the given equation when x tends to √6

Re-writing the equation as

$$= \lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{(\sqrt{3}+\sqrt{2})^2}}{x^2-6}$$

$$= \lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{3+2+2\sqrt{6}}}{x^2-6}$$

$$= \lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{5+2\sqrt{6}}}{x^2-6}$$

Now rationalizing the above equation

$$= \lim_{x\to\sqrt{6}} \frac{\left(\sqrt{5+2x}-\sqrt{5+2\sqrt{6}}\right)}{x^2-6} \frac{\left(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}}\right)}{\left(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}}\right)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x\to\sqrt{6}} \frac{\left(5+2x-\left(5+2\sqrt{6}\right)\right)}{x^2-6} \frac{(1)}{\left(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}}\right)}$$

$$= \lim_{x\to\sqrt{6}} \frac{\left(2x-2\sqrt{6}\right)}{x^2-6} \frac{(1)}{\left(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}}\right)}$$

$$= \lim_{x\to\sqrt{6}} \frac{2(x-\sqrt{6})}{(x+\sqrt{6})(x-\sqrt{6})} \frac{(1)}{\left(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}}\right)}$$

$$= \lim_{x \to \sqrt{6}} \frac{2(1)}{(x+\sqrt{6})(1)} \frac{(1)}{\left(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as √6

$$\lim_{x \to \sqrt{6}} \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6} = \frac{2}{2\sqrt{6}} \frac{1}{\left(2\sqrt{5+2\sqrt{6}}\right)} = \frac{1}{2\sqrt{6}} \frac{1}{\left(\sqrt{5+2\sqrt{6}}\right)}$$

34. Question

Evaluate the following limits:

$$\lim_{x \to \sqrt{2}} \frac{\sqrt{3+2x}-(\sqrt{2}+1)}{x^2-2}$$

Answer

Given $\lim_{x \to \sqrt{6}} \frac{\sqrt{3+2x}-(\sqrt{2}+\sqrt{1})}{x^2-2}$

To find: the limit of the given equation when x tends to $\sqrt{2}$

Re-writing the equation as

$$= \lim_{x \to \sqrt{2}} \frac{\sqrt{3+2x}-\sqrt{(\sqrt{2}+\sqrt{1})^2}}{x^2-2}$$

$$= \lim_{x \to \sqrt{2}} \frac{\sqrt{3+2x}-\sqrt{2+1+2\sqrt{2}}}{x^2-2}$$

$$= \lim_{x \to \sqrt{2}} \frac{\sqrt{3+2x}-\sqrt{3+2\sqrt{2}}}{x^2-2}$$

Now rationalizing the above equation

$$= \lim_{x \to \sqrt{2}} \frac{\left(\sqrt{3+2x}-\sqrt{3+2\sqrt{2}}\right)}{x^2-2} \frac{\left(\sqrt{3+2x}+\sqrt{3+2\sqrt{2}}\right)}{\left(\sqrt{3+2x}+\sqrt{3+2\sqrt{2}}\right)}$$

Formula: (a + b) (a - b) = a^2 - b^2

$$= \lim_{x \to \sqrt{2}} \frac{\left(3+2x-\left(3+2\sqrt{2}\right)\right)}{x^2-2} \frac{(1)}{\left(\sqrt{3+2x}+\sqrt{3+2\sqrt{2}}\right)}$$

$$= \lim_{x \to \sqrt{2}} \frac{\left(2x-2\sqrt{2}\right)}{x^2-2} \frac{(1)}{\left(\sqrt{3+2x}+\sqrt{3+2\sqrt{2}}\right)}$$

$$= \lim_{x \to \sqrt{2}} \frac{2\left(x-\sqrt{2}\right)}{(x+\sqrt{2})(x-\sqrt{2})} \frac{(1)}{\left(\sqrt{3+2x}+\sqrt{3+2\sqrt{2}}\right)}$$

$$= \lim_{x \to \sqrt{2}} \frac{2(1)}{(x+\sqrt{2})(1)} \frac{(1)}{\left(\sqrt{3+2x}+\sqrt{3+2\sqrt{2}}\right)}$$

Now we can see that the indeterminant form is removed, so substituting x as $\sqrt{2}$

$$\Rightarrow \lim_{x \to \sqrt{6}} \frac{\sqrt{3+2x}-(\sqrt{2}+\sqrt{1})}{x^2-2} = \frac{2}{2\sqrt{2}} \frac{1}{\left(2\sqrt{3+2\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}} \frac{1}{\left(\sqrt{3+2\sqrt{2}}\right)}$$

Exercise 29.5

1. Question

Evaluate the following limits:

$$\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x - a}$$

Answer

We need to find the limit for: $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$

As limit can’t be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

As Z does not match exactly with the form as described above so we need to do some manipulations–

Z = $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$

⇒ Z = $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x+2-(a+2)}$

Let x + 2 = y and a+2 = k

As x → a ; y → k

∴ Z = $\lim_{y \to k} \frac{(y)^{5/2} - (k)^{5/2}}{y-k}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

∴ Z = $\frac{5}{2} k^{\frac{5}{2}-1} = \frac{5}{2} k^{\frac{3}{2}} = \frac{5}{2} (a+2)^{\frac{3}{2}}$

Hence, $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a} = \frac{5}{2} (a+2)^{\frac{3}{2}}$

2. Question

Evaluate the following limits:

$$\lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x - a}$$

Answer

We need to find the limit for: $\lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

As limit can’t be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z does not match exactly with the form as described above so we need to do some manipulations–

$Z = \lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

$\Rightarrow Z = \lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x+2-(a+2)}$

Let x + 2 = y and a+2 = k

As x → a ; y → k

$\therefore Z = \lim_{y \to k} \frac{(y)^{3/2} - (k)^{3/2}}{y-k}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$\therefore Z = \frac{3}{2} k^{\frac{3}{2}-1} = \frac{3}{2} k^{\frac{1}{2}} = \frac{3}{2}(a+2)^{\frac{1}{2}}$

Hence, $\lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a} = \frac{3}{2}\sqrt{a+2}$

3. Question

Evaluate the following limits:

$$\lim_{x \to a} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$$

Answer

We need to find the limit for: $\lim_{x \to a} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

As limit can be find out simply by putting x = a because it is not taking indeterminate form(0/0) form, so we will be putting x = a

Let, $Z = \lim_{x \to a} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

$\Rightarrow Z = \frac{(1+a)^6 - 1}{(1+a)^2 - 1} = \frac{\{(1+a)^2\}^3 - 1}{(1+a)^2 - 1}$

This can be further simplified using $a^3 - 1 = (a-1)(a^2 + a + 1)$

$\Rightarrow Z = \frac{\{(1+a)^2 - 1\}((1+a)^4 + (1+a)^2 + 1)}{(1+a)^2 - 1}$

$\Rightarrow Z = (1+a)^4 + (1+a)^2 + 1$

4. Question

Evaluate the following limits:

$$\lim_{x \to a} \frac{x^{2/7} - a^{2/7}}{x - a}$$

Answer

We need to find the limit for: $\lim_{x \to a} \frac{x^{2/7} - a^{2/7}}{x-a}$

As limit can’t be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x \to a} \frac{x^{2/7} - a^{2/7}}{x - a}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z matches exactly with the form as described above so we don't need to do any manipulations–

Z = $\lim_{x \to a} \frac{x^{2/7} - a^{2/7}}{x - a}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$\therefore$ Z = $\frac{2}{7} a^{\frac{2}{7} - 1} = \frac{2}{7} a^{-\frac{5}{7}}$

Hence, $\lim_{x \to a} \frac{x^{2/7} - a^{2/7}}{x - a} = \frac{2}{7} a^{-\frac{5}{7}}$

5. Question

Evaluate the following limits:

$$\lim_{x \to a} \frac{x^{5/7} - a^{5/7}}{x^{2/7} - a^{2/7}}$$

Answer

We need to find the limit for: $\lim_{x \to a} \frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x^{\frac{2}{7}} - a^{\frac{2}{7}}}$

As limit can't be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x \to a} \frac{x^{5/7} - a^{5/7}}{x^{2/7} - a^{2/7}}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z does not match exactly with the form as described above so we need to do some manipulations–

Z = $\lim_{x \to a} \frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x^{\frac{2}{7}} - a^{\frac{2}{7}}}$

Dividing numerator and denominator by (x–a),we get

Z = $\lim_{x \to a} \frac{\frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x - a}}{\frac{x^{\frac{2}{7}} - a^{\frac{2}{7}}}{x - a}}$

Using algebra of limits, we have –

Z = $\frac{\lim_{x \to a} \frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x - a}}{\lim_{x \to a} \frac{x^{\frac{2}{7}} - a^{\frac{2}{7}}}{x - a}}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$\therefore Z = \frac{\frac{5}{7}a^{\frac{5}{7}-1}}{\frac{2}{7}a^{\frac{2}{7}-1}} = \frac{5a^{-\frac{2}{7}}}{2a^{-\frac{5}{7}}} = \frac{5}{2}a^{\frac{3}{7}}$

Hence, $\lim_{x \to a} \frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x^{\frac{2}{7}} - a^{\frac{2}{7}}} = \frac{5}{2}a^{\frac{3}{7}}$

6. Question

Evaluate the following limits:

$$\lim_{x \to -1/2} \frac{8x^3 + 1}{2x + 1}$$

Answer

We need to find the limit for: $\lim_{x \to -1/2} \frac{8x^3+1}{2x+1}$

As limit can't be find out simply by putting x = (–1/2) because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x \to -\frac{1}{2}} \frac{8x^3+1}{2x+1}$

⇒ Z = $\lim_{x \to -\frac{1}{2}} \frac{(2x)^3-(-1)}{2x-(-1)}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n-(a)^n}{x-a} = na^{n-1}$

As Z matches exactly with the form as described above so we don't need to do any manipulations–

Z = $\lim_{x \to -\frac{1}{2}} \frac{(2x)^3-(-1)^3}{2x-(-1)}$

Let y = 2x

As x → –1/2 ⇒ 2x = y → –1

$\therefore$ Z = $\lim_{y \to -1} \frac{y^3-(-1)^3}{y-(-1)}$

Use the formula: $\lim_{x \to a} \frac{(x)^n-(a)^n}{x-a} = na^{n-1}$

$\therefore$ Z = $3\,(-1)^{3-1} = 3(-1)^2 = 3$

Hence, $\lim_{x \to -\frac{1}{2}} \frac{8x^3+1}{2x+1} = 3$

7. Question

Evaluate the following limits:

$$\lim_{x \to 27} \frac{(x^{1/3} + 3)(x^{1/3} - 3)}{x - 27}$$

Answer

We need to find the limit for: $\lim_{x \to 27} \frac{(x^{1/3}+3)(x^{1/3}-3)}{x-27}$

As limit can't be find out simply by putting x = 27 because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x\to 27}\frac{(x^{1/3}+3)(x^{1/3}-3)}{x-27}$

Using algebra of limits, we have–

Z = $\lim_{x\to 27}\left(x^{\frac{1}{3}}+3\right)\times\lim_{x\to 27}\frac{(x^{1/3}-3)}{x-27}$

⇒ Z = $(27^{1/3}+3)\times\lim_{x\to 27}\frac{(x^{1/3}-3)}{x-27}$

⇒ Z = $6\lim_{x\to 27}\frac{(x^{1/3}-3)}{x-27}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a}=na^{n-1}$

As Z matches exactly with the form as described above so we don't need to do any manipulations–

Z = $6\lim_{x\to 27}\frac{x^{\frac{1}{3}}-(27)^{\frac{1}{3}}}{x-27}$

Use the formula: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a}=na^{n-1}$

∴ Z =$6\times\frac{1}{3}(27)^{\frac{1}{3}-1}=2\times(27)^{-\frac{2}{3}}=2\times3^{-2}=\frac{2}{9}$

Hence, $\lim_{x\to 27}\frac{(x^{1/3}+3)(x^{1/3}-3)}{x-27}=\frac{2}{9}$

8. Question

Evaluate the following limits:

$$\lim_{x\to 4}\frac{x^3-64}{x^2-16}$$

Answer

We need to find the limit for: $\lim_{x\to 4}\frac{x^3-64}{x^2-16}$

As limit can't be find out simply by putting x = 4 because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x\to 4}\frac{x^3-64}{x^2-16}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a}=na^{n-1}$

As Z does not match exactly with the form as described above so we need to do some manipulations–

Z = $\lim_{x\to 4}\frac{x^3-64}{x^2-16}=\lim_{x\to 4}\frac{x^3-4^3}{x^2-4^2}$

Dividing numerator and denominator by (x–4),we get

Z = $\lim_{x\to 4}\frac{\frac{x^3-4^3}{x-4}}{\frac{x^2-4^2}{x-4}}$

Using algebra of limits, we have –

$Z = \frac{\lim_{x \to 4} \frac{x^3 - 4^3}{x - 4}}{\lim_{x \to 4} \frac{x^2 - 4^2}{x - 4}}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$\therefore Z = \frac{3 \times (4)^{3-1}}{2 \times (4)^{2-1}} = \frac{3 \times 16}{2 \times 4} = 6$

Hence, $\lim_{x \to 4} \frac{x^3 - 64}{x^2 - 16} = 6$

9. Question

Evaluate the following limits:

$$\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1}$$

Answer

We need to find the limit for: $\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1}$

As limit can't be find out simply by putting x = 1 because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, $Z = \lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z does not match exactly with the form as described above so we need to do some manipulations–

$Z = \lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1} = \lim_{x \to 1} \frac{x^{15} - 1^{15}}{x^{10} - 1^{10}}$

Dividing numerator and denominator by (x–1),we get

$Z = \lim_{x \to 1} \frac{\frac{x^{15} - 1^{15}}{x - 1}}{\frac{x^{10} - 1^{10}}{x - 1}}$

Using algebra of limits, we have –

$Z = \frac{\lim_{x \to 1} \frac{x^{15} - 1^{15}}{x - 1}}{\lim_{x \to 1} \frac{x^{10} - 1^{10}}{x - 1}}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$\therefore Z = \frac{15 \times (1)^{15-1}}{10 \times (1)^{10-1}} = \frac{15}{10} = \frac{3}{2}$

Hence, $\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1} = \frac{3}{2}$

10. Question

Evaluate the following limits:

$$\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$$

Answer

We need to find the limit for: $\lim_{x\to -1}\frac{x^3+1}{x+1}$

As limit can't be find out simply by putting x = –1 because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x\to -1}\frac{x^3+1}{x+1}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a}=na^{n-1}$

As Z does matches exactly with the form as described above so we don't need to do any manipulations–

Z = $\lim_{x\to -1}\frac{x^3+1}{x+1}=\lim_{x\to -1}\frac{x^3-(-1)^3}{x-(-1)}$

Use the formula: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a}=na^{n-1}$

∴ Z = $3(-1)^{3-1}$ = 3

Hence, $\lim_{x\to -1}\frac{x^3+1}{x+1}=3$

11. Question

Evaluate the following limits:

$$\lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x^{3/4}-a^{3/4}}$$

Answer

We need to find the limit for: $\lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x^{3/4}-a^{3/4}}$

As limit can't be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x\to a}\frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x^{\frac{3}{4}}-a^{\frac{3}{4}}}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a}=na^{n-1}$

As Z does not match exactly with the form as described above so we need to do some manipulations–

Z = $\lim_{x\to a}\frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x^{\frac{3}{4}}-a^{\frac{3}{4}}}$

Dividing numerator and denominator by (x–a),we get

Z = $\lim_{x\to a}\frac{\frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x-a}}{\frac{x^{\frac{3}{4}}-a^{\frac{3}{4}}}{x-a}}$

Using algebra of limits, we have –

Z = $\frac{\lim_{x\to a}\frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x-a}}{\lim_{x\to a}\frac{x^{\frac{3}{4}}-a^{\frac{3}{4}}}{x-a}}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

$\therefore Z = \frac{\frac{2}{3} \times (a)^{\frac{2}{3}-1}}{\frac{3}{4} \times (a)^{\frac{3}{4}-1}} = \frac{\frac{2}{3}(a)^{-\frac{1}{3}}}{\frac{3}{4}(a)^{-\frac{1}{4}}} = \frac{8}{9}(a)^{-\frac{1}{3}+\frac{1}{4}} = \frac{8}{9} a^{-\frac{1}{12}}$

Hence, $\lim_{x \to a} \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{x^{\frac{3}{4}} - a^{\frac{3}{4}}} = \frac{8}{9} a^{-\frac{1}{12}}$

12. Question

If $\lim_{x \to 3} \frac{x^n - 3^n}{x-3} = 108$, find the value of n.

Answer

Given,

$\lim_{x \to 3} \frac{x^n - 3^n}{x-3} = 108$, we need to find value of n

So we will first find the limit and then equate it with 108 to get the value of n.

We need to find the limit for: $\lim_{x \to 3} \frac{x^n - 3^n}{x-3}$

As limit can't be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x \to 3} \frac{x^n - 3^n}{x-3}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

As Z matches exactly with the form as described above so we don't need to do any manipulations–

Z = $\lim_{x \to 3} \frac{x^n - 3^n}{x-3}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

$\therefore Z = n(3)^{n-1}$

According to question Z = 108

$\therefore n(3)^{n-1} = 108$

To solve such equations, factorize the number into prime factors and try to make combinations such that one satisfies with the equation.

$\Rightarrow n(3)^{n-1} = 4\times 27 = 4\times (3)^{4-1}$

Clearly on comparison we have –

n = 4

13. Question

if $\lim_{x \to a} \frac{x^9 - a^9}{x-a} = 9$, find all possible values of a.

Answer

Given,

$\lim_{x \to a} \frac{x^9 - a^9}{x - a} = 9$, we need to find value of n

So we will first find the limit and then equate it with 9 to get the value of n.

We need to find the limit for: $\lim_{x \to a} \frac{x^9 - a^9}{x - a}$

As limit can't be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x \to a} \frac{x^9 - a^9}{x - a}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z matches exactly with the form as described above so we don't need to do any manipulations–

Z = $\lim_{x \to a} \frac{x^9 - a^9}{x - a}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

∴ Z = $9(a)^{9-1} = 9a^8$

According to question Z = 9

∴ $9(a)^8 = 9$

⇒ $a^8 = 1 = 1^8$ or $(-1)^8$

Clearly on comparison we have –

a = 1 or –1

14. Question

If $\lim_{x \to a} \frac{x^5 - a^5}{x - a} = 405$, find all possible values of a.

Answer

Given,

$\lim_{x \to a} \frac{x^5 - a^5}{x - a} = 405$, we need to find value of n

So we will first find the limit and then equate it with 405 to get the value of n.

We need to find the limit for: $\lim_{x \to a} \frac{x^5 - a^5}{x - a}$

As limit can't be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, Z = $\lim_{x \to a} \frac{x^5 - a^5}{x - a}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z matches exactly with the form as described above so we don't need to do any manipulations–

$Z = \lim_{x \to a} \frac{x^5 - a^5}{x - a}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$\therefore Z = 5(a)^{5-1} = 5a^4$

According to question Z = 405

$\therefore 5(a)^4 = 405$

$\Rightarrow a^4 = 81 = 3^4$ or $(-3)^4$

Clearly on comparison we have –

a = 3 or –3

15. Question

If $\lim_{x \to a} \frac{x^9 - a^9}{x - a} = \lim_{x \to 5} (4 + x)$, find all possible values of a.

Answer

Given,

$\lim_{x \to a} \frac{x^9 - a^9}{x - a} = \lim_{x \to 5} (4 + x)$, we need to find value of n

So we will first find the limit and then equate it with $\lim_{x \to 5} (4 + x)$ to get the value of n.

We need to find the limit for: $\lim_{x \to a} \frac{x^9 - a^9}{x - a}$

As limit can't be find out simply by putting x = a because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, $Z = \lim_{x \to a} \frac{x^9 - a^9}{x - a}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z matches exactly with the form as described above so we don't need to do any manipulations–

$Z = \lim_{x \to a} \frac{x^9 - a^9}{x - a}$

Use the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$\therefore Z = 9(a)^{9-1} = 9a^8$

According to question $Z = \lim_{x \to 5} (4 + x) = 4 + 5 = 9$

$\therefore 9(a)^8 = 9$

$\Rightarrow a^8 = 1 = 1^8$ or $(-1)^8$

Clearly on comparison we have –

a = 1 or –1

16. Question

If $\lim_{x\to a}\frac{x^3-a^3}{x-a}=\lim_{x\to 1}\frac{x^4-1}{x-1}$, find all possible values of a.

Answer

Given,

$$\lim_{x\to a}\frac{x^3-a^3}{x-a}=\lim_{x\to 1}\frac{x^4-1}{x-1}$$

$$\Rightarrow \lim_{x\to a}\frac{x^3-a^3}{x-a}=\lim_{x\to 1}\frac{x^4-1^4}{x-1}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a}=na^{n-1}$

Using the formula we have –

$3a^{3-1} = 4(1)^{4-1}$

$\Rightarrow 3a^2 = 4$

$\Rightarrow a^2 = 4/3$

$\therefore a = \pm (2/\sqrt{3})$

Exercise 29.6

1. Question

Evaluate the following limits:

$$\lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}$$

Answer

Given: $\lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}$

$$\Rightarrow \lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}=\lim_{x\to\infty}\frac{(12x^2-10x+2)}{(x^2+9x-8)}$$

$$\Rightarrow \lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}=\lim_{x\to\infty}\left(\frac{12-\frac{10}{x}+\frac{2}{x^2}}{1+\frac{9}{x}-\frac{8}{x^2}}\right)$$

$x\to\infty$ and $\frac{1}{x}\to 0$ then,

$$\Rightarrow \lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}=\frac{12-0+0}{1}$$

Hence, $\lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}=12$

2. Question

Evaluate the following limits:

$$\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}$$

Answer

Given: $\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}$

$$\Rightarrow \lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}=\lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{2+\frac{1}{x}-\frac{5}{x^2}+\frac{7}{x^3}}$$

Since, $x\to\infty$ and $\frac{1}{x}\to 0$ then

$$\Rightarrow \lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}=\frac{3-0+0-0}{2+0-0+0}$$

Hence, $\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}=\frac{3}{2}$

3. Question

Evaluate the following limits:

$$\lim_{x\to\infty}\frac{5x^3-6}{\sqrt{9+4x^6}}$$

Answer

Given: $\lim_{x\to\infty}\frac{5x^3-6}{\sqrt{(9+4x^6)}}$

$$\Rightarrow \lim_{x\to\infty}\frac{5x^3-6}{\sqrt{(9+4x^6)}}=\lim_{x\to\infty}\frac{5-\frac{6}{x^3}}{\sqrt{(\frac{9}{x^6}+\frac{4x^6}{x^6}}}$$

$$\Rightarrow \lim_{x\to\infty}\frac{5x^3-6}{\sqrt{(9+4x^6)}}=\lim_{x\to\infty}\frac{\left(5-\frac{6}{x^3}\right)}{\sqrt{\frac{9}{x^6}+4}}$$

$$\Rightarrow \lim_{x\to\infty}\frac{5x^3-6}{\sqrt{(9+4x^6)}}=\frac{5}{\sqrt{4}}$$

Hence, $\lim_{x\to\infty}\frac{5x^3-6}{\sqrt{(9+4x^6)}}=\frac{5}{2}$

4. Question

Evaluate the following limits:

$$\lim_{x\to\infty}\sqrt{x^2+cx}-x$$

Answer

Given: $\lim_{x\to\infty}\sqrt{x^2+cx}-x$

Rationalizing the numerator we get,

$$\Rightarrow \lim_{x\to\infty}\sqrt{x^2+cx}-x=\lim_{x\to\infty}\left(\sqrt{x^2+cx}-x\right).\frac{\sqrt{x^2+cx}+x}{\sqrt{x^2+cx}+x}$$

$$\Rightarrow \lim_{x\to\infty}\sqrt{x^2+cx}-x=\lim_{x\to\infty}\frac{(x^2+cx-x^2)}{\sqrt{x^2+cx}+x}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+cx} - x = \lim_{x\to\infty} \frac{cx}{\sqrt{x^2+cx}+x}$$

Taking x common from both numerator and denominator

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+cx} - x = \lim_{x\to\infty} \frac{c}{\sqrt{1+\frac{c}{x}}+1}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+cx} - x = \frac{c}{1+1}$$

Hence, $\lim_{x\to\infty} \sqrt{x^2+cx} - x = \frac{c}{2}$

5. Question

Evaluate the following limits:

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$$

Answer

Given: $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$

On rationalizing the numerator we get,

$$\Rightarrow \lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}.\frac{\sqrt{x+1}+\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} \frac{(x+1-x)}{\sqrt{x+1}+\sqrt{x}}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} \left(\frac{1}{\sqrt{x+1}+\sqrt{x}}\right)$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \frac{1}{\infty}$$

Hence, $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = 0$

6. Question

Evaluate the following limits:

$$\lim_{x\to\infty} \sqrt{x^2+7x} - x$$

Answer

Given: $\lim_{x\to\infty} \sqrt{x^2+7x} - x$

On rationalizing the numerator we get,

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+7x} - x = \lim_{x\to\infty} \left(\sqrt{x^2+7x} - x\right).\frac{\sqrt{x^2+7x}+x}{\sqrt{x^2+7x}+x}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+7x} - x = \lim_{x\to\infty} \frac{(x^2+7x-x^2)}{\sqrt{x^2+7x}+x}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+7x} - x = \lim_{x\to\infty} \frac{7x}{\sqrt{x^2+7x}+x}$$

Taking x common from both numerator and denominator

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+7x} - x = \lim_{x\to\infty} \frac{7x}{\sqrt{\frac{x^2}{x^2} + \frac{7x}{x^2}} + 1}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+7x} - x = \lim_{x\to\infty} \frac{7x}{\sqrt{1 + \frac{7x}{x}} + 1}$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x^2+7x} - x = \frac{7}{\sqrt{1 + \frac{7}{x}} + 1}$$

Hence, $\lim_{x\to\infty} \sqrt{x^2+7x} - x = \frac{7}{2}$

7. Question

Evaluate the following limits:

$$\lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1} - 1}$$

Answer

Given: $\lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1}-1}$

$$\Rightarrow \lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1} - 1} = \lim_{x\to\infty} \frac{1}{\sqrt{4 + \frac{1}{x^2}} - \frac{1}{x}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1} - 1} = \lim_{x\to\infty} \frac{1}{\sqrt{4 + \frac{1}{\infty}} - \frac{1}{\infty}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1} - 1} = \frac{1}{\sqrt{4}}$$

Hence, $\lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1}-1} = \frac{1}{2}$

8. Question

Evaluate the following limits:

$$\lim_{n\to\infty} \frac{n^2}{1+2+3+\ldots+n}$$

Answer

Given: $\lim_{x\to\infty} \frac{n^2}{1+2+3+\cdots+n}$

We know that,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

By putting this Formula, we get,

$$\Rightarrow \lim_{x\to\infty} \frac{n^2}{1+2+3+\cdots+n} = \lim_{x\to\infty} \frac{n^2}{\frac{1}{2}n(n+1)}$$

$$\Rightarrow \lim_{x\to\infty}\frac{n^2}{1+2+3+\cdots+n}=\lim_{x\to\infty}\frac{2n^2}{n^2+n}$$

$$\Rightarrow \lim_{x\to\infty}\frac{n^2}{1+2+3+\cdots+n}=2.\lim_{x\to\infty}\frac{n^2}{n^2+n}$$

$$\Rightarrow \lim_{x\to\infty}\frac{n^2}{1+2+3+\cdots+n}=2.\lim_{x\to\infty}\frac{n^2}{n^2\left(1+\frac{1}{n}\right)}$$

$$\Rightarrow \lim_{x\to\infty}\frac{n^2}{1+2+3+\cdots+n}=2.\frac{1}{1+0}$$

Hence, $\lim_{x\to\infty}\frac{n^2}{1+2+3+\cdots+n}=2$

9. Question

Evaluate the following limits:

$$\lim_{x\to\infty}\frac{3x^{-1}+4x^{-2}}{5x^{-1}+6x^{-2}}$$

Answer

Given: $\lim_{x\to\infty}\frac{3x^{-1}+4x^{-2}}{5x^{-1}+6x^{-2}}$

$$\Rightarrow \lim_{x\to\infty}\frac{3x^{-1}+4x^{-2}}{5x^{-1}+6x^{-2}}=\lim_{x\to\infty}\frac{\frac{1}{x}\left(3+\frac{4}{x}\right)}{\frac{1}{x}\left(5+\frac{6}{x}\right)}$$

$$\Rightarrow \lim_{x\to\infty}\frac{3x^{-1}+4x^{-2}}{5x^{-1}+6x^{-2}}=\lim_{x\to\infty}\frac{3+0}{5+0}$$

Hence, $\lim_{x\to\infty}\frac{3x^{-1}+4x^{-2}}{5x^{-1}+6x^{-2}}=\frac{3}{5}$

10. Question

Evaluate the following limits:

$$\lim_{x\to\infty}\frac{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}+\sqrt{x^2+d^2}}$$

Answer

Given: $\lim_{x\to\infty}\frac{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}+\sqrt{x^2+d^2}}$

$$\Rightarrow \lim_{x\to\infty}\frac{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}+\sqrt{x^2+d^2}}\left[\frac{\infty}{\infty}\text{ form}\right]$$

Rationalizing the numerator and denominator we get,

$$\Rightarrow \lim_{x\to\infty}\frac{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}+\sqrt{x^2+d^2}}\times\frac{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}$$

$$\Rightarrow \lim_{x\to\infty}\frac{((x^2+a^2)-(x^2-b^2))}{(\sqrt{x^2+c^2}+\sqrt{x^2+d^2})\sqrt{x^2+a^2}-\sqrt{x^2+b^2})}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(a^2-b^2)}{(\sqrt{x^2+c^2}+\sqrt{x^2+d^2})\sqrt{x^2+a^2}-\sqrt{x^2+b^2})}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(a^2-b^2)(\sqrt{x^2+c^2}+\sqrt{x^2+d^2})}{(\sqrt{x^2+c^2}+\sqrt{x^2+d^2})(\sqrt{x^2+c^2}-\sqrt{x^2+d^2})\sqrt{x^2+a^2}-\sqrt{x^2+b^2})}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(a^2-b^2)(\sqrt{x^2+c^2}+\sqrt{x^2+d^2})}{(x^2+c^2-x^2-d^2)(\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(a^2-b^2)(\sqrt{x^2+c^2}+\sqrt{x^2+d^2})}{(c^2-d^2)(\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(a^2-b^2)\left(x\sqrt{1+\frac{c^2}{x^2}}+x\sqrt{1+\frac{d^2}{x^2}}\right)}{(c^2-d^2)\left(x\sqrt{1+\frac{a^2}{x^2}}+x\sqrt{1+\frac{b^2}{x^2}}\right)}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(a^2-b^2)\left(\sqrt{1+\frac{c^2}{x^2}}+\sqrt{1+\frac{d^2}{x^2}}\right)}{(c^2-d^2)\left(\sqrt{1+\frac{a^2}{x^2}}+\sqrt{1+\frac{b^2}{x^2}}\right)}$$

$$\Rightarrow \frac{(a^2-b^2)(1+1)}{(c^2-d^2)(1+1)}$$

Hence, $\lim_{x\to\infty} \frac{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}+\sqrt{x^2+d^2}} = \frac{(a^2-b^2)}{(c^2-d^2)}$

11. Question

Evaluate the following limits:

$$\lim_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$$

Answer

Given: $\lim_{x\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!}$

We know that,

(n + 2)! = (n + 2) × (n + 1)!

By putting the value of (n+2)! , we get

$$\Rightarrow \lim_{x\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!} = \lim_{x\to\infty} \frac{(n+2)(n+1)!+(n+1)!}{(n+2)(n+1)!-(n+1)!}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!} = \lim_{x\to\infty} \frac{(n+1)!\,[(n+2)+1]}{(n+1)!\,[(n+2)-1]}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!} = \lim_{x\to\infty} \frac{n+2+1}{n+2-1}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!} = \lim_{x\to\infty} \frac{n+3}{n+1}$$

$$\Rightarrow \lim_{x\to\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!}=\lim_{x\to\infty}\frac{n\left(1+\frac{3}{n}\right)}{n\left(1+\frac{1}{n}\right)}$$

$$\Rightarrow \lim_{x\to\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!}=\lim_{x\to\infty}\frac{1+\frac{3}{n}}{1+\frac{1}{n}}$$

$$\Rightarrow \lim_{x\to\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!}=\frac{1+0}{1+0}$$

Hence, $\lim_{x\to\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n-1)!}=1$

12. Question

Evaluate the following limits:

$$\lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}$$

Answer

Given: $\lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}$

On Rationalizing the Numerator we get,

$$\Rightarrow \lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}$$
$$=\lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}\times\frac{x\sqrt{x^2+1}+\sqrt{x^2-1}}{x\sqrt{x^2+1}+\sqrt{x^2+1}}$$

$$\Rightarrow \lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}=\lim_{x\to\infty} x\frac{x(x^2+1-x^2+1)}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$\Rightarrow \lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}=\lim_{x\to\infty} x\frac{2x}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$\Rightarrow \lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}=\lim_{x\to\infty}\frac{2x^2}{x^2\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}}$$

$$\Rightarrow \lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}=\lim_{x\to\infty}\frac{2x^2}{x^2\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}}$$

$$\Rightarrow \lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}=\lim_{x\to\infty}\frac{2x^2}{x^2\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}}$$

$$\Rightarrow \lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}}$$

$$\Rightarrow \lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}=\frac{2}{1+1}$$

Hence, $\lim_{x\to\infty} x\left\{\sqrt{x^2+1}-\sqrt{x^2-1}\right\}=1$

13. Question

Evaluate the following limits:

$$\lim_{x\to\infty} x\left\{\sqrt{x+1}-\sqrt{x}\right\}\sqrt{x+2}$$

Answer

Given: $\lim_{x\to\infty} x\{\sqrt{x+1}-\sqrt{x}\}\sqrt{x+2}$

On Rationalizing the numerator we get,

$$\Rightarrow \lim_{x\to\infty}\{\sqrt{x+1}-\sqrt{x}\}\sqrt{x+2} = \lim_{x\to\infty}\frac{[\{\sqrt{x+1}-\sqrt{x}\}\sqrt{x+2}\{\sqrt{x+1}+\sqrt{x}\}]}{\{\sqrt{x+1}+\sqrt{x}\}}$$

$$\Rightarrow \lim_{x\to\infty} x\{\sqrt{x+1}-\sqrt{x}\}\sqrt{x+2} = \lim_{x\to\infty}\frac{(\sqrt{x+2})(x+1-x)}{\{\sqrt{x+1}+\sqrt{x}\}}$$

Dividing the numerator and the denominator by √x, we get,

$$\Rightarrow \lim_{x\to\infty}\{\sqrt{x+1}-\sqrt{x}\}\sqrt{x+2} = \lim_{x\to\infty}\frac{\frac{(\sqrt{x+2})}{\sqrt{x}}}{\frac{\{\sqrt{x+1}+\sqrt{x}\}}{\sqrt{x}}}$$

$$\Rightarrow \lim_{x\to\infty}\{\sqrt{x+1}-\sqrt{x}\}\sqrt{x+2} = \lim_{x\to\infty}\frac{\sqrt{1+\frac{2}{x}}}{\sqrt{1+\frac{1}{x}}+1}$$

$$\Rightarrow \lim_{x\to\infty}\{\sqrt{x+1}-\sqrt{x}\}\sqrt{x+2} = .\frac{1}{\sqrt{1}+1}$$

Hence, $\lim_{x\to\infty}\{\sqrt{x+1}-\sqrt{x}\}\sqrt{x+2} = \frac{1}{2}$

14. Question

Evaluate the following limits:

$$\lim_{n\to\infty}\frac{1^2+2^2+\ldots.+n^2}{n^3}$$

Answer

Given: $\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}$

Formula Used:

$$\Rightarrow 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Now, Putting this formula and we get,

$$\Rightarrow \lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3} = \lim_{n\to\infty}\frac{1}{6}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{6}\lim_{n\to\infty}\left[\frac{(n^2+n)(2n+1)}{n^3}\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{6}\lim_{n\to\infty}\left[\frac{(2n^3+n^2+2n^2+n)}{n^3}\right]$$

Taking x^3 as common and we get,

$$\Rightarrow \lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{6}\lim_{n\to\infty}\frac{n^3}{n^3}\left[\frac{\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{1}\right] \left(\frac{\infty}{\infty}\text{ form}\right)$$

Since, $n \to \infty$ and $\frac{1}{n} \to 0$ then,

$$\Rightarrow \lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{6}\cdot\frac{2+0+0}{1} = \frac{1}{3}$$

Hence, $\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{1}{3}$

15. Question

Evaluate the following limits:

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots.+\frac{n-1}{n^2}\right)$$

Answer

Given: $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n-1}{n^2}\right)$

Taking LCM then, we get,

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n-1}{n^2}\right) = \lim_{n\to\infty}\left(\frac{1+2+3+\cdots+(n-1)}{n^2}\right)$$

Therefore,

$$\left[\frac{1+2+3+\cdots+(n-1)}{n^2} = \frac{(n-1)n}{2n^2}\right]$$

By putting this, we get,

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n-1}{n^2}\right) = \lim_{n\to\infty}\left(\frac{(n-1)(n)}{2n^2}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n-1}{n^2}\right) = \lim_{n\to\infty}\left(\frac{n^2-n}{2n^2}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n-1}{n^2}\right) = \lim_{n\to\infty}\frac{n^2}{n^2}\left(\frac{1-\frac{1}{n}}{2}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n-1}{n^2}\right) = \frac{1-0}{2} = \frac{1}{2}$$

Hence, $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n-1}{n^2}\right) = \frac{1}{2}$

16. Question

Evaluate the following limits:

$$\lim_{n\to\infty}\frac{1^3+2^3+\ldots+n^3}{n^4}$$

Answer

Given: $\lim_{n\to\infty}\frac{1^3+2^3+\cdots+n^3}{n^4}$

Here we know that,

$$\Rightarrow 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \lim_{n\to\infty} \frac{\left[\frac{1}{2}n(n+1)\right]^2}{n^4}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \lim_{n\to\infty} \frac{\frac{1}{4}n^2(n+1)^2}{n^4}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \lim_{n\to\infty} \frac{1}{4} \cdot \frac{n^2(n^2+1+2n)}{n^4}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \lim_{n\to\infty} \frac{1}{4} \cdot \frac{n^4+n^2+2n}{n^4}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \lim_{n\to\infty} \frac{1}{4} \cdot \frac{n^4}{n^4}\left[1 + \frac{1}{n^2} + \frac{2}{n}\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \frac{1}{4} \lim_{n\to\infty} \left[1 + \frac{1}{n^2} + \frac{2}{n}\right]$$

Since, $n \to \infty$ and $\frac{1}{n} \to 0$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \frac{1}{4}\,[1+0+0]$$

Hence, $\lim_{n\to\infty} \frac{1^3+2^3+\cdots+n^3}{n^4} = \frac{1}{4}$

17. Question

Evaluate the following limits:

$$\lim_{n\to\infty} \frac{1^3 + 2^3 + \ldots + n^3}{(n-1)^4}$$

Answer

Formula Used:

$$\Rightarrow 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Given: $\lim_{n\to\infty} \frac{1^3+2^3+\cdots+n^3}{(n-1)^4}$

By putting this, in the given equation, we get,

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{(n-1)^4} = \lim_{n\to\infty} \frac{\left[\frac{1}{2}.n.(n+1)\right]^2}{(n-1)^4}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{(n-1)^4} = \lim_{n\to\infty} \left[\frac{\frac{1}{4}n^2(n^2+1+2n)}{(n-1)^4}\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{(n-1)^4} = \frac{1}{4} . \lim_{n\to\infty} \left[\frac{n^4+n^2+2n^3}{(n-1)^2(n-1)^2}\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3+2^3+\cdots+n^3}{(n-1)^4} = \frac{1}{4}.\lim_{n\to\infty}\left[\frac{n^4+n^2+2n^3}{(n^2+1-2n)(n^2+1-2n)}\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^3+2^3+\cdots+n^3}{(n-1)^4}$$
$$= \frac{1}{4}.\lim_{n\to\infty}\left[\frac{n^4+n^2+2n^3}{n^4+n^2-2n^3+n^2+1-2n-2n^3-2n+4n^2}\right]$$

Taking x^4 as common,

$$\Rightarrow \lim_{n\to\infty} \frac{1^3+2^3+\cdots+n^3}{(n-1)^4}$$
$$= \frac{1}{4}.\lim_{n\to\infty}\frac{n^4}{n^4}\left[\frac{\left(1+\frac{1}{n^2}+\frac{2}{n}\right)}{1+\frac{1}{n^2}-\frac{2}{n}+\frac{1}{n^2}+\frac{1}{n^4}-\frac{2}{n^3}-\frac{2}{n}-\frac{2}{n^3}+\frac{4}{n^2}}\right]$$
$$\Rightarrow \lim_{n\to\infty} \frac{1^3+2^3+\cdots+n^3}{(n-1)^4} = \frac{1}{4}.\left(\frac{1}{1}\right)$$

Hence, $\lim_{n\to\infty} \frac{1^3+2^3+\cdots+n^3}{(n-1)^4} = \frac{1}{4}$

18. Question

Evaluate the following limits:

$$\lim_{x\to\infty} \sqrt{x}\left\{\sqrt{x+1}-\sqrt{x}\right\}$$

Answer

$$\Rightarrow \lim_{x\to\infty} \sqrt{x}\{\sqrt{x+1}-\sqrt{x}\} = \lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$$

Now, Rationalizing the Numerator, we get,

$$\Rightarrow \lim_{x\to\infty} \sqrt{x}\{\sqrt{x+1}-\sqrt{x}\} = \lim_{x\to\infty}\left[\sqrt{x^2+x}-x \times \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right]$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x}\{\sqrt{x+1}-\sqrt{x}\} = \lim_{x\to\infty}\left[\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}\right]$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x}\{\sqrt{x+1}-\sqrt{x}\} = \lim_{x\to\infty}\left[\frac{x}{\sqrt{x^2+x}+x}\right]$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x}\{\sqrt{x+1}-\sqrt{x}\} = \lim_{x\to\infty}\left[\frac{1}{\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}}+1}\right]$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x}\{\sqrt{x+1}-\sqrt{x}\} = \lim_{x\to\infty}\left[\frac{1}{\sqrt{1+\frac{1}{x}}+1}\right]$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x}\{\sqrt{x+1}-\sqrt{x}\} = \left[\frac{1}{1+1}\right]$$

Hence, $\lim_{x\to\infty} \sqrt{x}\{\sqrt{x+1}-\sqrt{x}\} = \frac{1}{2}$

19. Question

Evaluate the following limits:

$$\lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+....+\frac{1}{3^n}\right)$$

Answer

$\lim_{n\to\infty}\left[\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^n}\right]$ ......(1)

We can see that this is a geometric progression with the common ratio 1/3.

And, we know the sum of n terms of GP is $S_n = a\left[\frac{1-r^n}{1-r}\right]$

Let suppose, $a = \frac{1}{3}$ and $r = \frac{1}{3}$, then

$$S_n = \frac{1}{3}\left[\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}\right]$$

$$= \frac{1}{3}\left[\frac{\left(1-\frac{1}{3^n}\right)}{\frac{2}{3}}\right]$$

$$= \frac{1}{3}\times\frac{3}{2}\left[1-\frac{1}{3^n}\right]$$

$$S_n = \frac{1}{2}\left[1-\frac{1}{3^n}\right]$$

Now, putting the value of S_n in equation (1), we get

$$\Rightarrow \lim_{n\to\infty}\left[\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^n}\right] = \frac{1}{2}\lim_{n\to\infty}\left[1-\frac{1}{3^n}\right]$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^n}\right] = \frac{1}{2}(1-0)$$

Hence, $\lim_{n\to\infty}\left[\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^n}\right] = \frac{1}{2}$

20. Question

Evaluate the following limits:

$\lim_{x\to\infty}\frac{x^4+7x^3+46x+a}{x^4+6}$, where a is a non-zero real number.

Answer

Give: $\lim_{x\to\infty}\frac{x^4+7x^3+46x+a}{x^4+6}$

Now, Taking x^4 as common from both numerator and denominator,

$$\Rightarrow \lim_{x\to\infty}\frac{x^4+7x^3+46x+a}{x^4+6} = \lim_{x\to\infty}\frac{x^4}{x^4}\left[\frac{1+\frac{7}{x}+\frac{46}{x^3}+\frac{a}{x^4}}{1+\frac{6}{x^4}}\right]$$

$$\Rightarrow \lim_{x\to\infty}\frac{x^4+7x^3+46x+a}{x^4+6} = \frac{1+\frac{a}{0}}{1}$$

$$\Rightarrow \lim_{x\to\infty}\frac{x^4+7x^3+46x+a}{x^4+6} = \frac{0+a}{1}$$

Hence, a = 1

21. Question

Evaluate the following limits:

$f(x)=\frac{ax^2+b}{x^2+1}, \lim_{x\to 0} f(x)=1$ and $\lim_{x\to\infty} f(x)=1$, then prove that f(-2) = f(2) = 1.

Answer

Given: $f(x)=\frac{ax^2+b}{x^2+1}, \lim_{x\to 0} f(x)=1$ and $\lim_{x\to\infty} f(x)=1$

To Prove: f(-2) = f(2) = 1.

Proof: we have, $f(x)=\frac{ax^2+b}{x^2+1}$

And, $\lim_{x\to 0} f(x)=1$

$$\Rightarrow \lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{ax^2+b}{x^2+1}=1$$

$$\Rightarrow \lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{ax^2+b}{x^2+1}=\frac{\lim_{x\to 0} ax^2+b}{\lim_{x\to 0} x^2+1}$$

Therefore, b = 1

Also, $\lim_{x\to\infty} f(x)=1$

$$\Rightarrow \lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{ax^2+b}{x^2+1}=1$$

$$\Rightarrow \lim_{x\to\infty} f(x)=\frac{\lim_{x\to 0} ax^2+b}{\lim_{x\to 0} x^2+1}$$

b = 1

Thus, $f(x)=\frac{ax^2+b}{x^2+1}$

On substituting the value of a and b we get,

$$f(x)=\frac{ax^2+b}{x^2+1}=\frac{x^2+1}{x^2+1}$$

$$\Rightarrow \lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^2+1}{x^2+1}$$

So, f(x) = 1

Then, f(-2) = 1

Also, f(2) = 1

Hence, f(2)=f(-2)=1

22. Question

Show that $\lim_{x\to\infty}(\sqrt{x^2+x+1}-x)\neq\lim_{x\to\infty}(\sqrt{x^2+1}-x)$

To Prove: $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)\neq\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

Answer

We have L.H.S = $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)$

Rationalizing the numerator, we get,

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)\times\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)=\lim_{x\to\infty}\frac{(x^2+x+1-x^2)}{\sqrt{x^2+x+1}+x}$$

Taking x as common from both numerator and denominator,

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)=\lim_{x\to\infty}\frac{x\left(1+\frac{1}{x}\right)}{x\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1\right]}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)=\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)=\frac{1}{1+1}$$

Therefore, $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)=\frac{1}{2}$

Now , Take R.H.S $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\sqrt{x^2+1}-x\times\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{x^2+1-x^2}{x\sqrt{1+\frac{1}{x^2}}+x}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{1}{x\sqrt{1+\frac{1}{x^2}}+1}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{1}{x}.\frac{1}{\sqrt{1+\frac{1}{x^2}}+1}$$

Now $x\to\infty$ and $\frac{1}{x}=0$ then

Therefore, R.H.S = 0

So, L.H.S ≠ R.H.S

Hence, $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)\neq\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

23. Question

Evaluate the following limits:

$$\lim_{x\to-\infty}\left(\sqrt{4x^2-7x}+2x\right)$$

Answer

Rationalizing the numerator, we get

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{4x^2-7x}+2x\right)=\lim_{x\to\infty}\left(\sqrt{4x^2-7x}+2x\right)\times\frac{\sqrt{4x^2-7x}-2x}{\sqrt{4x^2-7x}-2x}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{4x^2-7x}+2x\right)=\lim_{x\to\infty}\frac{4x^2-7x-4x^2}{\sqrt{4x^2-7x}-2x}$$

Taking x as common from both numerator and denominator,

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{4x^2-7x}+2x\right)=\lim_{x\to\infty}\frac{-7}{\left[\sqrt{4-\frac{7}{x}-\frac{1}{x}}\right]}$$

Now $x\to\infty$ and $\frac{1}{x}=0$ then

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{4x^2-7x}+2x\right)=-\frac{7}{1}=-7$$

Hence, $\lim_{x\to\infty}\left(\sqrt{4x^2-7x}+2x\right)=-7.$

24. Question

Evaluate the following limits:

$$\lim_{x\to-\infty}\left(\sqrt{x^2-8x}+x\right)$$

Answer

Rationalizing the numerator, we get

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2-8x}+x\right)=\lim_{x\to\infty}\left(\sqrt{x^2-8x}+x\right)\times\frac{\sqrt{x^2-8x}-x}{\sqrt{x^2-8x}-x}$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2-8x}+x\right)=\lim_{x\to\infty}\frac{(-8x)}{\sqrt{x^2-8x}-x}$$

Taking x as common from both numerator and denominator,

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2-8x}+x\right)=\lim_{x\to\infty}\frac{-8}{\left[\sqrt{1-\frac{8}{x}-\frac{1}{x}}\right]}$$

Now $x\to\infty$ and $\frac{1}{x}=0$ then

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x^2-8x}+x\right)=-\frac{8}{1}=-8$$

Hence, $\lim_{x\to\infty}\left(\sqrt{x^2-8x}+x\right)=-8.$

25. Question

Evaluate:

$$\lim_{n\to\infty}\frac{1^4+2^4+3^4+\ldots+n^4}{n^5}-\lim_{n\to\infty}\frac{1^3+2^3+\ldots+n^3}{\underset{x\to\infty}{n^5}}$$

Answer

Formula Used:

$$\Rightarrow 1^4+2^4+3^4+\cdots+n^4=\frac{n(1+n)(1+2n)(-1+3n+3n^2)}{30}$$

$$\Rightarrow 1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Now putting these value, we get,

$$\Rightarrow \lim_{n\to\infty}\frac{\left(\frac{n(1+n)(1+2n)(-1+3n+3n^2)}{30}\right)}{n^5}-\lim_{n\to\infty}\frac{\left(\left(\frac{n(n+1)}{2}\right)^2\right)}{n^5}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)\left(-\frac{1}{n^2}+\frac{3}{n}+3\right)}{30}-\lim_{n\to\infty}\frac{1}{n^5}\left(\frac{n^2(n^2+2n+1)}{4}\right)$$

$$\Rightarrow \lim_{n\to\infty}\frac{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)\left(-\frac{1}{n^2}+\frac{3}{n}+3\right)}{30}-\lim_{n\to\infty}\left(\frac{\frac{1}{n}+\frac{2}{n^2}+\frac{1}{n^3}}{4}\right)$$

Now $n\to\infty$ and $\frac{1}{n}=0$ then,

$$=\frac{1\times 2\times 3}{30}-0$$

$$=\frac{1}{5}$$

26. Question

Evaluate:

$$\lim_{n\to\infty}\frac{1.2+2.3+3.4+\ldots+n(n+1)}{n^3}$$

Answer

Here We know,

$$\Rightarrow 1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow 1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

By putting these value, we get,

$$\Rightarrow \lim_{n\to\infty}\frac{1.2+2.3+3.4+\cdots+n(n-1)}{n^3}$$

$$=\lim_{n\to\infty}\frac{\frac{(n(n+1)(2n+1))}{6}+\frac{n(n+1)}{2}}{n^3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1.2+2.3+3.4+\cdots+n(n-1)}{n^3} = \lim_{n\to\infty} \frac{\frac{(n(n+1)(2n+1)+3n(n+1))}{6}}{n^3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1.2+2.3+3.4+\cdots+n(n-1)}{n^3} = \lim_{n\to\infty} \frac{n(n+1)\left[\frac{(2n+1)+3}{6}\right]}{n^3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1.2+2.3+3.4+\cdots+n(n-1)}{n^3} = \lim_{n\to\infty} \frac{\frac{n(n+1)(2n+4)}{6}}{n^3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1.2+2.3+3.4+\cdots+n(n-1)}{n^3} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{4}{n}\right)}{6}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1.2+2.3+3.4+\cdots+n(n-1)}{n^3} = \frac{1\times 2}{6} = \frac{1}{3}$$

Hence, $\lim_{n\to\infty} \frac{1.2+2.3+3.4+\cdots+n(n-1)}{n^3} = \frac{1}{3}$

Exercise 29.7

1. Question

Evaluate the following limits:

$$\lim_{x\to 0} \frac{\sin 3x}{5x}$$

Answer

To find: $\lim_{x\to 0} \frac{\sin 3x}{5x}$

$$\lim_{x\to 0} \frac{\sin 3x}{5x}$$

$$= \frac{1}{5} \lim_{x\to 0} \frac{\sin 3x}{x}$$

Multiplying and Dividing by 3:

$$= \frac{1}{5} \lim_{x\to 0} \frac{\sin 3x}{3x} \times 3$$

$$= \frac{3}{5} \lim_{x\to 0} \frac{\sin 3x}{3x}$$

As, $x \to 0 \Rightarrow 3x \to 0$

$$= \frac{3}{5} \lim_{3x\to 0} \frac{\sin 3x}{3x}$$

Now, put $3x = y$

$$= \frac{3}{5} \lim_{y\to 0} \frac{\sin y}{y}$$

Formula used:

$$\lim_{y\to 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\lim_{x\to 0}\frac{\sin 3x}{5x}$$

$$=\frac{3}{5}\lim_{y\to 0}\frac{\sin y}{y}$$

$$=\frac{3}{5}\times 1$$

$$=\frac{3}{5}$$

Hence the value of $\lim_{x\to 0}\frac{\sin 3x}{5x}=\frac{3}{5}$

2. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin x^{0}}{x}$$

Answer

To find: $\lim_{x\to 0}\frac{\sin x^{\circ}}{x}$

We know, $1^{\circ}=\frac{\pi}{180}$ radians

$\therefore x^{\circ}=\frac{\pi x}{180}$ radians

$$\lim_{x\to 0}\frac{\sin x^{\circ}}{x}$$

$$=\lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{x}$$

Multiplying and Dividing by $\frac{\pi}{180}$

$$=\lim_{x\to 0}\frac{\sin\frac{\pi x}{180}\times\frac{\pi}{180}}{x\times\frac{\pi}{180}}$$

$$=\frac{\pi}{180}\lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}$$

As, $x\to 0\Rightarrow\frac{\pi x}{180}\to 0$

$$=\frac{\pi}{180}\lim_{\frac{\pi x}{180}\to 0}\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}$$

Now, put $\frac{\pi x}{180}=y$

$$=\frac{\pi}{180}\lim_{y\to 0}\frac{\sin y}{y}$$

Formula used:

$$\lim_{y\to 0}\frac{\sin y}{y}=1$$

Therefore,

$$\lim_{x\to 0}\frac{\sin x^{\circ}}{x}$$

$$=\frac{\pi}{180}\lim_{y\to 0}\frac{\sin y}{y}$$

$$=\frac{\pi}{180}\times 1$$

$$=\frac{\pi}{180}$$

Hence, the value of $\lim_{x\to 0}\frac{\sin x^{\circ}}{x}=\frac{\pi}{180}$

3. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{x^2}{\sin x^2}$$

Answer

To find: $\lim_{x\to 0}\frac{x^2}{\sin x^2}$

$$\lim_{x\to 0}\frac{x^2}{\sin x^2}$$

$$=\lim_{x\to 0}\frac{1}{\frac{\sin x^2}{x^2}}$$

As, $x \to 0 \Rightarrow x^2 \to 0$

$$=\lim_{x^2\to 0}\frac{1}{\frac{\sin x^2}{x^2}}$$

$$=\frac{1}{\lim_{x^2\to 0}\frac{\sin x^2}{x^2}}$$

Now, put $x^2=y$

$$=\frac{1}{\lim_{y\to 0}\frac{\sin y}{y}}$$

Formula used:

$$\lim_{y\to 0}\frac{\sin y}{y}=1$$

Therefore,

$$\lim_{x\to 0}\frac{x^2}{\sin x^2}$$

$$= \frac{1}{\lim_{y \to 0} \frac{\sin y}{y}}$$

$$= \frac{1}{1}$$

=1

Hence, the value of $\lim_{x \to 0} \frac{x^2}{\sin x^2} = 1$

4. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sin x \cos x}{3x}$$

Answer

To find: $\lim_{x \to 0} \frac{\sin x \cos x}{3x}$

$$\lim_{x \to 0} \frac{\sin x \cos x}{3x}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{\sin x \cos x}{x}$$

$$= \frac{1}{3} \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \cos x$$

We know,

$$\lim_{x \to 0} A(x).B(x) = \lim_{x \to 0} A(x) \times \lim_{x \to 0} B(x)$$

Therefore,

$$= \frac{1}{3} \lim_{x \to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \cos x$$

Formula used:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x \cos x}{3x}$$

$$= \frac{1}{3} \times 1 \times \cos 0$$

$$= \frac{1}{3} \times 1 \times 1$$

$\{\because \cos 0 = 1\}$

$$= \frac{1}{3}$$

Hence, the value of $\lim_{x \to 0} \frac{\sin x \cos x}{3x} = \frac{1}{3}$

5. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{3\sin x-4\sin^3 x}{x}$$

Answer

To find: $\lim_{x\to 0}\frac{3\sin x-4\sin^3 x}{x}$

We know,

Sin3x = 3sinx – 4 $\sin^3$x

Therefore,

$$\lim_{x\to 0}\frac{3\sin x-4\sin^3 x}{x}$$

$$=\lim_{x\to 0}\frac{\sin 3x}{x}$$

Multiplying and Dividing by 3:

$$=\lim_{x\to 0}\frac{\sin 3x\times 3}{3x}$$

$$=3\lim_{x\to 0}\frac{\sin 3x}{3x}$$

As, $x \to 0 \Rightarrow 3x \to 0$

$$=3\lim_{3x\to 0}\frac{\sin 3x}{3x}$$

Now, put 3x = y

$$=3\lim_{y\to 0}\frac{\sin y}{y}$$

Formula used:

$$\lim_{y\to 0}\frac{\sin y}{y}=1$$

Therefore,

$$\lim_{x\to 0}\frac{3\sin x-4\sin^3 x}{x}$$

$$=3\lim_{y\to 0}\frac{\sin y}{y}$$

= 3 × 1

= 3

Hence, the value of $\lim_{x\to 0}\frac{3\sin x-4\sin^3 x}{x}=3$

6. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\tan 8x}{\sin 2x}$$

Answer

To find: $\lim_{x\to 0}\frac{\tan 8x}{\sin 2x}$

$\lim_{x\to 0}\frac{\tan 8x}{\sin 2x}$

Multiplying and Dividing by 8x in numerator & Multiplying and Dividing by 2x in the denominator:

$$= \lim_{x\to 0}\frac{\frac{\tan 8x}{8x}\times 8x}{\frac{\sin 2x}{2x}\times 2x}$$

$$= \lim_{x\to 0}\frac{\frac{\tan 8x}{8x}}{\frac{\sin 2x}{2x}}\times\frac{8x}{2x}$$

$$= \lim_{x\to 0}\frac{\frac{\tan 8x}{8x}}{\frac{\sin 2x}{2x}}\times 4$$

We know,

$$\lim_{x\to 0}\frac{A(x)}{B(x)} = \frac{\lim_{x\to 0}A(x)}{\lim_{x\to 0}B(x)}$$

Therefore,

$$= 4\times\frac{\lim_{x\to 0}\frac{\tan 8x}{8x}}{\lim_{x\to 0}\frac{\sin 2x}{2x}}$$

As, x → 0 ⇒ 8x → 0 & 2x → 0

$$= 4\times\frac{\lim_{8x\to 0}\frac{\tan 8x}{8x}}{\lim_{2x\to 0}\frac{\sin 2x}{2x}}$$

Now, put 2x = y and 8x = t

$$= 4\times\frac{\lim_{t\to 0}\frac{\tan t}{t}}{\lim_{y\to 0}\frac{\sin y}{y}}$$

Formula used:

$$\lim_{y\to 0}\frac{\sin y}{y} = 1 \;\&\; \lim_{t\to 0}\frac{\tan t}{t} = 1$$

Therefore,

$$\lim_{x\to 0}\frac{\tan 8x}{\sin 2x}$$

$$= 4\times\frac{\lim_{t\to 0}\frac{\tan t}{t}}{\lim_{y\to 0}\frac{\sin y}{y}}$$

$$= 4\times\frac{1}{1}$$

$= 4$

Hence, the value of $\lim_{x\to 0}\frac{\tan 8x}{\sin 2x} = 4$

7. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\tan\ m\,x}{\tan nx}$$

Answer

To find: $\lim_{x\to 0}\frac{\tan mx}{\tan nx}$

$$\lim_{x\to 0}\frac{\tan mx}{\tan nx}$$

Multiplying and Dividing by mx in numerator & Multiplying and Dividing by nx in the denominator:

$$= \lim_{x\to 0}\frac{\frac{\tan mx}{mx}\times mx}{\frac{\tan nx}{nx}\times nx}$$

$$= \lim_{x\to 0}\frac{\frac{\tan mx}{mx}}{\frac{\tan nx}{nx}}\times\frac{mx}{nx}$$

$$= \lim_{x\to 0}\frac{\frac{\tan mx}{mx}}{\frac{\tan nx}{nx}}\times\frac{m}{n}$$

We know,

$$\lim_{x\to 0}\frac{A(x)}{B(x)} = \frac{\lim_{x\to 0} A(x)}{\lim_{x\to 0} B(x)}$$

Therefore,

$$= \frac{m}{n}\times\frac{\lim_{x\to 0}\frac{\tan mx}{mx}}{\lim_{x\to 0}\frac{\tan nx}{nx}}$$

As, x → 0 ⇒ mx → 0 & nx → 0

$$= \frac{m}{n}\times\frac{\lim_{mx\to 0}\frac{\tan mx}{mx}}{\lim_{nx\to 0}\frac{\tan nx}{nx}}$$

Now, put mx = y and nx = t

$$= \frac{m}{n}\times\frac{\lim_{y\to 0}\frac{\tan y}{y}}{\lim_{t\to 0}\frac{\tan t}{t}}$$

Formula used:

$$\lim_{t\to 0}\frac{\tan t}{t} = 1$$

Therefore,

$\lim_{x\to 0}\frac{\tan mx}{\tan nx}$

$= \frac{m}{n}\times\frac{\lim_{y\to 0}\frac{\tan y}{y}}{\lim_{t\to 0}\frac{\tan t}{t}}$

$= \frac{m}{n}\times\frac{1}{1}$

$= \frac{m}{n}$

Hence, the value of $\lim_{x\to 0}\frac{\tan mx}{\tan nx} = \frac{m}{n}$

8. Question

Evaluate the following limits:

$\lim_{x\to 0}\frac{\sin 5x}{\tan 3x}$

Answer

To find: $\lim_{x\to 0}\frac{\sin 5x}{\tan 3x}$

$\lim_{x\to 0}\frac{\sin 5x}{\tan 3x}$

Multiplying and Dividing by 5x in numerator & Multiplying and Dividing by 3x in the denominator:

$= \lim_{x\to 0}\frac{\frac{\sin 5x}{5x}\times 5x}{\frac{\tan 3x}{3x}\times 3x}$

$= \lim_{x\to 0}\frac{\frac{\sin 5x}{5x}}{\frac{\tan 3x}{3x}}\times\frac{5x}{3x}$

$= \lim_{x\to 0}\frac{\frac{\sin 5x}{5x}}{\frac{\tan 3x}{3x}}\times\frac{5}{3}$

We know,

$\lim_{x\to 0}\frac{A(x)}{B(x)} = \frac{\lim_{x\to 0}A(x)}{\lim_{x\to 0}B(x)}$

Therefore,

$= \frac{5}{3}\times\frac{\lim_{x\to 0}\frac{\sin 5x}{5x}}{\lim_{x\to 0}\frac{\tan 3x}{3x}}$

As, $x \to 0 \Rightarrow 5x \to 0$ & $3x \to 0$

$= \frac{5}{3}\times\frac{\lim_{5x\to 0}\frac{\sin 5x}{5x}}{\lim_{3x\to 0}\frac{\tan 3x}{3x}}$

Now, put 5x = y and 3x = t

$$= \frac{5}{3} \times \frac{\lim_{y \to 0} \frac{\sin y}{y}}{\lim_{t \to 0} \frac{\tan t}{t}}$$

Formula used:

$$\lim_{y \to 0} \frac{\sin y}{y} = 1 \ \& \ \lim_{t \to 0} \frac{\tan t}{t} = 1$$

Therefore,

$$\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$$

$$= \frac{5}{3} \times \frac{\lim_{y \to 0} \frac{\sin y}{y}}{\lim_{t \to 0} \frac{\tan t}{t}}$$

$$= \frac{5}{3} \times \frac{1}{1}$$

$$= \frac{5}{3}$$

Hence, the value of $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x} = \frac{5}{3}$

9. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sin x^0}{x^0}$$

Answer

To find: $\lim_{x \to 0} \frac{\sin x°}{x°}$

We know, $1° = \frac{\pi}{180}$ radians

$$\therefore x° = \frac{\pi x}{180} \text{ radians}$$

$$\lim_{x \to 0} \frac{\sin x°}{x°}$$

$$= \lim_{x \to 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}}$$

$$\text{As}, x \to 0 \Rightarrow \frac{\pi x}{180} \to 0$$

$$= \lim_{\frac{\pi x}{180} \to 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}}$$

$$\text{Now, put } \frac{\pi x}{180} = y$$

$$= \lim_{y \to 0} \frac{\sin y}{y}$$

Formula used:

$$\lim_{y \to 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\lim_{x \to 0} \frac{\sin x^\circ}{x^\circ}$$

$$= \lim_{y \to 0} \frac{\sin y}{y}$$

= 1

Hence, the value of $\lim_{x \to 0} \frac{\sin x^\circ}{x^\circ} = 1$

10. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$$

Answer

To find: $\lim_{x \to 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$

$$\lim_{x \to 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$$

Dividing numerator and denominator by x:

$$= \lim_{x \to 0} \frac{\frac{7x \cos x - 3 \sin x}{x}}{\frac{4x + \tan x}{x}}$$

$$= \lim_{x \to 0} \frac{7 \cos x - \frac{3 \sin x}{x}}{4 + \frac{\tan x}{x}}$$

We know,

$$\lim_{x \to 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \to 0} A(x) - \lim_{x \to 0} B(x)}{\lim_{x \to 0} C(x) - \lim_{x \to 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \to 0} 7 \cos x - \lim_{x \to 0} \frac{3 \sin x}{x}}{\lim_{x \to 0} 4 + \lim_{x \to 0} \frac{\tan x}{x}}$$

Formula used:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \ \& \ \lim_{x \to 0} \frac{\tan x}{x} = 1$$

Therefore,

$$\lim_{x\to 0}\frac{7x\cos x - 3\sin x}{4x + \tan x}$$

$$= \frac{\lim_{x\to 0} 7\cos x - 3\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0} 4 + \lim_{x\to 0}\frac{\tan x}{x}}$$

$$= \frac{7\cos 0 - 3\times 1}{4+1}$$

$\{\because \cos 0 = 1\}$

$$= \frac{7-3}{5}$$

$$= \frac{4}{5}$$

Hence, the value of $\lim_{x\to 0}\frac{7x\cos x - 3\sin x}{4x+\tan x} = \frac{4}{5}$

11. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\cos ax - \cos bx}{\cos cx - \cos dx}$$

Answer

To find: $\lim_{x\to 0}\frac{\cos ax - \cos bx}{\cos cx - \cos dx}$

We know,

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Therefore,

$$\lim_{x\to 0}\frac{\cos ax - \cos bx}{\cos cx - \cos dx}$$

$$= \lim_{x\to 0}\frac{-2\sin\frac{ax+bx}{2}\sin\frac{ax-bx}{2}}{-2\sin\frac{cx+dx}{2}\sin\frac{cx-dx}{2}}$$

$$= \lim_{x\to 0}\frac{\sin\frac{(a+b)x}{2}\sin\frac{(a-b)x}{2}}{\sin\frac{(c+d)x}{2}\sin\frac{(c-d)x}{2}}$$

Multiplying and Dividing by $\frac{(a+b)x}{2}\times\frac{(a-b)x}{2}$ in numerator &

similarly by $\frac{(c+d)x}{2}\times\frac{(c-d)x}{2}$ in denominator, we get,

$$= \lim_{x\to 0}\frac{\left(\frac{\sin\frac{(a+b)x}{2}}{\frac{(a+b)x}{2}}\times\frac{(a+b)x}{2}\right)\left(\frac{\sin\frac{(a-b)x}{2}}{\frac{(a-b)x}{2}}\times\frac{(a-b)x}{2}\right)}{\left(\frac{\sin\frac{(c+d)x}{2}}{\frac{(c+d)x}{2}}\times\frac{(c+d)x}{2}\right)\left(\frac{\sin\frac{(c-d)x}{2}}{\frac{(c-d)x}{2}}\times\frac{(c-d)x}{2}\right)}$$

We know,

$$\lim_{x\to 0}\frac{A(x)\times B(x)}{C(x)\times D(x)}=\frac{\lim\limits_{x\to 0}A(x)\times\lim\limits_{x\to 0}B(x)}{\lim\limits_{x\to 0}C(x)\times\lim\limits_{x\to 0}D(x)}$$

Therefore,

$$=\frac{\lim\limits_{x\to 0}\left(\frac{\sin\frac{(a+b)x}{2}}{\frac{(a+b)x}{2}}\times\frac{(a+b)x}{2}\right)\times\lim\limits_{x\to 0}\left(\frac{\sin\frac{(a-b)x}{2}}{\frac{(a-b)x}{2}}\times\frac{(a-b)x}{2}\right)}{\lim\limits_{x\to 0}\left(\frac{\sin\frac{(c+d)x}{2}}{\frac{(c+d)x}{2}}\times\frac{(c+d)x}{2}\right)\times\lim\limits_{x\to 0}\left(\frac{\sin\frac{(c-d)x}{2}}{\frac{(c-d)x}{2}}\times\frac{(c-d)x}{2}\right)}$$

$$\text{As, } x\to 0\Rightarrow\frac{(a+b)x}{2}\to 0;\ \frac{(a-b)x}{2}\to 0;\ \frac{(c+d)x}{2}\to 0;\ \frac{(c-d)x}{2}\to 0$$

$$=\frac{\lim\limits_{\frac{(a+b)x}{2}\to 0}\left(\frac{\sin\frac{(a+b)x}{2}}{\frac{(a+b)x}{2}}\times\frac{(a+b)x}{2}\right)\times\lim\limits_{\frac{(a-b)x}{2}\to 0}\left(\frac{\sin\frac{(a-b)x}{2}}{\frac{(a-b)x}{2}}\times\frac{(a-b)x}{2}\right)}{\lim\limits_{\frac{(c+d)x}{2}\to 0}\left(\frac{\sin\frac{(c+d)x}{2}}{\frac{(c+d)x}{2}}\times\frac{(c+d)x}{2}\right)\times\lim\limits_{\frac{(c-d)x}{2}\to 0}\left(\frac{\sin\frac{(c-d)x}{2}}{\frac{(c-d)x}{2}}\times\frac{(c-d)x}{2}\right)}$$

$$\text{Put } \frac{(a+b)x}{2}=m\,;\ \frac{(a-b)x}{2}=n\,;\ \frac{(c+d)x}{2}=k\,;\ \frac{(c-d)x}{2}=l$$

$$=\frac{\lim\limits_{m\to 0}\left(\frac{\sin m}{m}\times m\right)\times\lim\limits_{n\to 0}\left(\frac{\sin n}{n}\times n\right)}{\lim\limits_{k\to 0}\left(\frac{\sin k}{k}\times k\right)\times\lim\limits_{l\to 0}\left(\frac{\sin l}{l}\times l\right)}$$

Formula used:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Therefore,

$$\lim_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-\cos dx}$$

$$=\frac{\lim\limits_{m\to 0}(1\times m)\times\lim\limits_{n\to 0}(1\times n)}{\lim\limits_{k\to 0}(1\times k)\times\lim\limits_{l\to 0}(1\times l)}$$

Now, put values of m, n, k and l:

$$=\frac{\lim\limits_{m\to 0}\left(\frac{(a+b)x}{2}\right)\times\lim\limits_{n\to 0}\left(\frac{(a-b)x}{2}\right)}{\lim\limits_{k\to 0}\left(\frac{(c+d)x}{2}\right)\times\lim\limits_{l\to 0}\left(\frac{(c-d)x}{2}\right)}$$

$$=\lim_{x\to 0}\frac{\left(\frac{(a+b)x}{2}\right)\left(\frac{(a-b)x}{2}\right)}{\left(\frac{(c+d)x}{2}\right)\left(\frac{(c-d)x}{2}\right)}$$

$$=\lim_{x\to 0}\frac{(a+b)(a-b)}{(c+d)(c-d)}$$

$= \frac{(a+b)(a-b)}{(c+d)(c-d)}$

$= \frac{a^2 - b^2}{c^2 - d^2}$

Hence, the value of $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} = \frac{a^2 - b^2}{c^2 - d^2}$

12. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\tan^2 3x}{x^2}$$

Answer

To find: $\lim_{x \to 0} \frac{\tan^2 3x}{x^2}$

$\lim_{x \to 0} \frac{\tan^2 3x}{x^2}$

$= \lim_{x \to 0} \left(\frac{\tan 3x}{x}\right)^2$

Multiplying and dividing by 3^2 :

$= \lim_{x \to 0} \left(\frac{\tan 3x}{x}\right)^2 \times \frac{3^2}{3^2}$

$= \lim_{x \to 0} \left(\frac{\tan 3x}{3x}\right)^2 \times 3^2$

Now, put $3x = y$

$= 3^2 \times \lim_{y \to 0} \left(\frac{\tan y}{y}\right)^2$

Formula used:

$\lim_{y \to 0} \frac{\tan y}{y} = 1$

Therefore,

$= 3^2 \times \lim_{y \to 0} \left(\frac{\tan y}{y}\right)^2$

$= 9 \times 1$

$= 9$

Hence, the value of $\lim_{x \to 0} \frac{\tan^2 3x}{x^2} = 9$

13. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{1 - \cos mx}{x^2}$$

Answer

To find: $\lim_{x\to 0}\frac{1-\cos mx}{x^2}$

We know,

cos 2x = 1 – 2 sin^2 x

$\Rightarrow \cos mx = 1 - 2\sin^2\frac{mx}{2}$

$\Rightarrow 1 - \cos mx = 2\sin^2\frac{mx}{2}$

$\lim_{x\to 0}\frac{1-\cos mx}{x^2}$

$= \lim_{x\to 0}\frac{2\sin^2\frac{mx}{2}}{x^2}$

$= 2\times\lim_{x\to 0}\left(\frac{\sin\frac{mx}{2}}{x}\right)^2$

Multiplying and dividing by $\left(\frac{m}{2}\right)^2$:

$= 2\times\lim_{x\to 0}\left(\frac{\sin\frac{mx}{2}}{x}\right)^2\times\frac{\left(\frac{m}{2}\right)^2}{\left(\frac{m}{2}\right)^2}$

$= 2\times\lim_{x\to 0}\left(\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right)^2\times\left(\frac{m}{2}\right)^2$

$\text{As}, x\to 0\Rightarrow\frac{mx}{2}\to 0$

$= 2\times\lim_{\frac{mx}{2}\to 0}\left(\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right)^2\times\frac{m^2}{4}$

$\text{Put}\frac{mx}{2} = y:$

$= \frac{2m^2}{4}\times\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2$

Formula used:

$\lim_{y\to 0}\frac{\sin y}{y} = 1$

Therefore,

$= \frac{m^2}{2}\times\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2$

$= \frac{m^2}{2}\times 1$

$$= \frac{m^2}{2}$$

Hence, the value of $\lim_{x \to 0} \frac{1 - \cos mx}{x^2} = \frac{m^2}{2}$

14. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$$

Answer

To find: $\lim_{x \to 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$

$$\lim_{x \to 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$$

Dividing numerator and denominator by 6x:

$$= \lim_{x \to 0} \frac{\frac{3 \sin 2x + 2x}{6x}}{\frac{3x + 2 \tan 3x}{6x}}$$

$$= \lim_{x \to 0} \frac{\frac{3 \sin 2x}{6x} + \frac{2x}{6x}}{\frac{3x}{6x} + \frac{2 \tan 3x}{6x}}$$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} + \frac{1}{3}}{\frac{1}{2} + \frac{\tan 3x}{3x}}$$

We know,

$$\lim_{x \to 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \to 0} A(x) - \lim_{x \to 0} B(x)}{\lim_{x \to 0} C(x) - \lim_{x \to 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \to 0} \frac{\sin 2x}{2x} + \lim_{x \to 0} \frac{1}{3}}{\lim_{x \to 0} \frac{1}{2} + \lim_{x \to 0} \frac{\tan 3x}{3x}}$$

As, x → 0 ⇒ 2x → 0 & 3x → 0

$$= \frac{\lim_{2x \to 0} \frac{\sin 2x}{2x} + \frac{1}{3}}{\frac{1}{2} + \lim_{3x \to 0} \frac{\tan 3x}{3x}}$$

Put 2x = y and 3x = k;

$$= \frac{\lim_{y \to 0} \frac{\sin y}{y} + \frac{1}{3}}{\frac{1}{2} + \lim_{k \to 0} \frac{\tan k}{k}}$$

Formula used:

$\lim_{y\to 0}\frac{\sin y}{y} = 1$ & $\lim_{k\to 0}\frac{\tan k}{k} = 1$

Therefore,

$$\lim_{x\to 0}\frac{3\sin 2x + 2x}{3x + 2\tan 3x}$$

$$= \frac{\lim_{y\to 0}\frac{\sin y}{y} + \frac{1}{3}}{\frac{1}{2} + \lim_{k\to 0}\frac{\tan k}{k}}$$

$$= \frac{1 + \frac{1}{3}}{\frac{1}{2} + 1}$$

$$= \frac{\frac{3+1}{3}}{\frac{1+2}{2}}$$

$$= \frac{\frac{4}{3}}{\frac{3}{2}}$$

$$= \frac{4}{3} \times \frac{2}{3}$$

$$= \frac{8}{9}$$

Hence, the value of $\lim_{x\to 0}\frac{3\sin 2x+2x}{3x+2\tan 3x} = \frac{8}{9}$

15. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\cos 3x - \cos 7x}{x^2}$$

Answer

To find: $\lim_{x\to 0}\frac{\cos 3x - \cos 7x}{x^2}$

We know,

$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$$

Therefore,

$$\lim_{x\to 0}\frac{\cos 3x - \cos 7x}{x^2}$$

$$= \lim_{x\to 0}\frac{2\sin\frac{3x+7x}{2}\sin\frac{7x-3x}{2}}{x^2}$$

$$= \lim_{x\to 0}\frac{2\sin\frac{10x}{2}\sin\frac{4x}{2}}{x^2}$$

$$= 2 \times \lim_{x \to 0} \frac{\sin 5x}{x} \times \frac{\sin 2x}{x}$$

Multiplying and dividing by 10:

$$= 2 \times 10 \times \lim_{x \to 0} \frac{\sin 5x}{5x} \times \frac{\sin 2x}{2x}$$

As,

x → 0 ⇒ 2x → 0 & 5x → 0

$$\lim_{x \to 0} A(x) \times B(x) = \lim_{x \to 0} A(x) \times \lim_{x \to 0} B(x)$$

$$= 20 \times \lim_{5x \to 0} \frac{\sin 5x}{5x} \times \lim_{2x \to 0} \frac{\sin 2x}{2x}$$

Put 2x = y and 5x = k;

$$= 20 \times \lim_{k \to 0} \frac{\sin k}{k} \times \lim_{y \to 0} \frac{\sin y}{y}$$

Formula used:

$$\lim_{y \to 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\lim_{x \to 0} \frac{\cos 3x - \cos 7x}{x^2}$$

$$= 20 \times \lim_{k \to 0} \frac{\sin k}{k} \times \lim_{y \to 0} \frac{\sin y}{y}$$

=20 × 1

= 20

Hence, the value of $\lim_{x \to 0} \frac{\cos 3x - \cos 7x}{x^2} = 20$

16. Question

Evaluate the following limits:

$$\lim_{\theta \to 0} \frac{\sin 3\theta}{\tan 2\theta}$$

Answer

To find: $\lim_{x \to 0} \frac{\sin 3\theta}{\tan 2\theta}$

$$\lim_{x \to 0} \frac{\sin 3\theta}{\tan 2\theta}$$

Multiplying and Dividing by 3θ in numerator & Multiplying and Dividing by 2θ in the denominator:

$$= \lim_{x \to 0} \frac{\frac{\sin 3\theta}{3\theta} \times 3\theta}{\frac{\tan 2\theta}{2\theta} \times 2\theta}$$

$$= \lim_{x\to 0} \frac{\frac{\sin 3\theta}{3\theta}}{\frac{\tan 2\theta}{2\theta}} \times \frac{3\theta}{2\theta}$$

$$= \lim_{x\to 0} \frac{\frac{\sin 3\theta}{3\theta}}{\frac{\tan 2\theta}{2\theta}} \times \frac{3}{2}$$

We know,

$$\lim_{x\to 0} \frac{A(x)}{B(x)} = \frac{\lim_{x\to 0} A(x)}{\lim_{x\to 0} B(x)}$$

Therefore,

$$= \frac{3}{2} \times \frac{\lim_{x\to 0} \frac{\sin 3\theta}{3\theta}}{\lim_{x\to 0} \frac{\tan 2\theta}{2\theta}}$$

As, x → 0 ⇒ 3θ → 0 & 2θ → 0

$$= \frac{3}{2} \times \frac{\lim_{3\theta\to 0} \frac{\sin 3\theta}{3\theta}}{\lim_{2\theta\to 0} \frac{\tan 2\theta}{2\theta}}$$

Now, put 3θ = y and 2θ = t

$$= \frac{3}{2} \times \frac{\lim_{y\to 0} \frac{\sin y}{y}}{\lim_{t\to 0} \frac{\tan t}{t}}$$

Formula used:

$$\lim_{y\to 0} \frac{\sin y}{y} = 1 \ \& \ \lim_{t\to 0} \frac{\tan t}{t} = 1$$

Therefore,

$$\lim_{x\to 0} \frac{\sin 3\theta}{\tan 2\theta}$$

$$= \frac{3}{2} \times \frac{\lim_{y\to 0} \frac{\sin y}{y}}{\lim_{t\to 0} \frac{\tan t}{t}}$$

$$= \frac{3}{2} \times \frac{1}{1}$$

$$= \frac{3}{2}$$

Hence, the value of $\lim_{x\to 0} \frac{\sin 3\theta}{\tan 2\theta} = \frac{3}{2}$

17. Question

Evaluate the following limits:

$$\lim_{x\to 0} \frac{\sin x^2 (1-\cos x^2)}{x^6}$$

Answer

To find: $\lim_{x\to 0}\frac{\sin x^2(1-\cos x^2)}{x^6}$

We know,

cos 2x = 1 – 2 $\sin^2$ x

$\Rightarrow \cos x^2 = 1 - 2\sin^2\frac{x^2}{2}$

$\Rightarrow 1 - \cos x^2 = 2\sin^2\frac{x^2}{2}$

$\lim_{x\to 0}\frac{\sin x^2(1-\cos x^2)}{x^6}$

$= \lim_{x\to 0}\frac{\sin x^2}{x^2}\times\frac{1-\cos x^2}{x^4}$

$= \lim_{x\to 0}\frac{\sin x^2}{x^2}\times\frac{2\sin^2\frac{x^2}{2}}{x^4}$

$= 2\times\lim_{x\to 0}\frac{\sin x^2}{x^2}\times\left(\frac{\sin\frac{x^2}{2}}{x^2}\right)^2\times\frac{\frac{1}{4}}{\frac{1}{4}}$

$= 2\times\lim_{x\to 0}\frac{\sin x^2}{x^2}\times\left(\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\right)^2\times\frac{1}{4}$

$= \frac{2}{4}\times\lim_{x\to 0}\frac{\sin x^2}{x^2}\times\left(\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\right)^2$

$\lim_{x\to 0} A(x)\times B(x) = \lim_{x\to 0} A(x)\times\lim_{x\to 0} B(x)$

$= \frac{1}{2}\times\lim_{x\to 0}\frac{\sin x^2}{x^2}\times\left(\lim_{x\to 0}\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\right)^2$

$\text{As}, x\to 0\Rightarrow x^2\to 0\ \&\ \frac{x^2}{2}\to 0$

$= \frac{1}{2}\times\lim_{x^2\to 0}\frac{\sin x^2}{x^2}\times\left(\lim_{\frac{x^2}{2}\to 0}\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\right)^2$

$\text{Put } x^2 = y;\ \frac{x^2}{2} = t$

$= \frac{1}{2}\times\lim_{y\to 0}\frac{\sin y}{y}\times\left(\lim_{t\to 0}\frac{\sin t}{t}\right)^2$

Formula used:

$\lim_{y\to 0}\frac{\sin y}{y} = 1$

Therefore,

$$= \frac{1}{2} \times \lim_{y\to0} \frac{\sin y}{y} \times \left(\lim_{t\to0} \frac{\sin t}{t}\right)^2$$

$$= \frac{1}{2} \times 1$$

$$= \frac{1}{2}$$

Hence, the value of $\lim_{x\to0} \frac{\sin x^2 (1-\cos x^2)}{x^6} = \frac{1}{2}$

18. Question

Evaluate the following limits:

$$\lim_{x\to0} \frac{\sin^2 4x^2}{x^4}$$

Answer

To find: $\lim_{x\to0} \frac{\sin^2 4x^2}{x^4}$

$$\lim_{x\to0} \frac{\sin^2 4x^2}{x^4}$$

$$= \lim_{x\to0} \left(\frac{\sin 4x^2}{x^2}\right)^2 \times \frac{16}{16}$$

$$= \lim_{x\to0} \left(\frac{\sin 4x^2}{4x^2}\right)^2 \times 16$$

$$= 16 \times \lim_{x\to0} \left(\frac{\sin 4x^2}{4x^2}\right)^2$$

As, $x \to 0 \Rightarrow x^2 \to 0 \Rightarrow 4x^2 \to 0$

$$= 16 \times \lim_{4x^2\to0} \left(\frac{\sin 4x^2}{4x^2}\right)^2$$

Put $x^2 = y$

$$= 16 \times \lim_{y\to0} \left(\frac{\sin y}{y}\right)^2$$

Formula used:

$$\lim_{y\to0} \frac{\sin y}{y} = 1$$

Therefore,

$= 16 \times (1)^2$

$= 16$

Hence, the value of $\lim_{x\to0} \frac{\sin^2 4x^2}{x^4} = 16$

19. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$$

Answer

To find: $\lim_{x \to 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$

$$\lim_{x \to 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$$

Dividing numerator and denominator by x:

$$= \lim_{x \to 0} \frac{\frac{x \cos x + 2 \sin x}{x}}{\frac{x^2 + \tan x}{x}}$$

$$= \lim_{x \to 0} \frac{\cos x + \frac{2 \sin x}{x}}{x + \frac{\tan x}{x}}$$

We know,

$$\lim_{x \to 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \to 0} A(x) - \lim_{x \to 0} B(x)}{\lim_{x \to 0} C(x) - \lim_{x \to 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \to 0} \cos x + \lim_{x \to 0} \frac{2 \sin x}{x}}{\lim_{x \to 0} x + \lim_{x \to 0} \frac{\tan x}{x}}$$

Formula used:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \;\&\; \lim_{x \to 0} \frac{\tan x}{x} = 1$$

Therefore,

$$\lim_{x \to 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$$

$$= \frac{\lim_{x \to 0} \cos x + 2 \lim_{x \to 0} \frac{\sin x}{x}}{\lim_{x \to 0} x + \lim_{x \to 0} \frac{\tan x}{x}}$$

$$= \frac{\cos 0 + 2 \times 1}{0 + 1}$$

$\{\because \cos 0 = 1\}$

$$= \frac{1 + 2}{1}$$

$$= \frac{3}{1}$$

$= 3$

Hence, the value of $\lim_{x \to 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x} = 3$

20. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{2x - \sin x}{\tan x + x}$$

Answer

To find: $\lim_{x \to 0} \frac{2x - \sin x}{\tan x + x}$

$$\lim_{x \to 0} \frac{2x - \sin x}{\tan x + x}$$

Dividing numerator and denominator by x:

$$= \lim_{x \to 0} \frac{\frac{2x - \sin x}{x}}{\frac{\tan x + x}{x}}$$

$$= \lim_{x \to 0} \frac{\frac{2x}{x} - \frac{\sin x}{x}}{\frac{\tan x}{x} + \frac{x}{x}}$$

$$= \lim_{x \to 0} \frac{2 - \frac{\sin x}{x}}{\frac{\tan x}{x} + 1}$$

We know,

$$\lim_{x \to 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \to 0} A(x) - \lim_{x \to 0} B(x)}{\lim_{x \to 0} C(x) - \lim_{x \to 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \to 0} 2 - \lim_{x \to 0} \frac{\sin x}{x}}{\lim_{x \to 0} \frac{\tan x}{x} + \lim_{x \to 0} 1}$$

Formula used:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \ \& \ \lim_{x \to 0} \frac{\tan x}{x} = 1$$

Therefore,

$$\lim_{x \to 0} \frac{2x - \sin x}{\tan x + x}$$

$$= \frac{2 - \lim_{x \to 0} \frac{\sin x}{x}}{\lim_{x \to 0} \frac{\tan x}{x} + 1}$$

$$= \frac{2 - 1}{1 + 1}$$

$$= \frac{1}{2}$$

Hence, the value of $\lim_{x \to 0} \frac{2x - \sin x}{\tan x + x} = \frac{1}{2}$

21. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{5x\cos x+3\sin x}{3x^2+\tan x}$$

Answer

To find: $\lim_{x\to 0}\frac{5x\cos x+3\sin x}{3x^2+\tan x}$

$$\lim_{x\to 0}\frac{5x\cos x+3\sin x}{3x^2+\tan x}$$

Dividing numerator and denominator by x:

$$=\lim_{x\to 0}\frac{\frac{5x\cos x+3\sin x}{x}}{\frac{3x^2+\tan x}{x}}$$

$$=\lim_{x\to 0}\frac{5\cos x+\frac{3\sin x}{x}}{3x+\frac{\tan x}{x}}$$

We know,

$$\lim_{x\to 0}\frac{A(x)-B(x)}{C(x)+D(x)}=\frac{\lim_{x\to 0}A(x)-\lim_{x\to 0}B(x)}{\lim_{x\to 0}C(x)-\lim_{x\to 0}D(x)}$$

Therefore,

$$=\frac{\lim_{x\to 0}5\cos x+\lim_{x\to 0}\frac{3\sin x}{x}}{\lim_{x\to 0}3x+\lim_{x\to 0}\frac{\tan x}{x}}$$

Formula used:

$$\lim_{x\to 0}\frac{\sin x}{x}=1\ \&\ \lim_{x\to 0}\frac{\tan x}{x}=1$$

Therefore,

$$\lim_{x\to 0}\frac{5x\cos x+3\sin x}{3x^2+\tan x}$$

$$=\frac{\lim_{x\to 0}5\cos x+3\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0}3x+\lim_{x\to 0}\frac{\tan x}{x}}$$

$$=\frac{5\cos 0+3\times 1}{3\times 0+1}$$

$\{\because \cos 0=1\}$

$$=\frac{5+3}{0+1}$$

$$=\frac{8}{1}$$

= 8

Hence, the value of $\lim_{x\to 0}\frac{5x\cos x+3\sin x}{3x^2+\tan x}=8$

22. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin 3x-\sin x}{\sin x}$$

Answer

To find: $\lim_{x\to 0}\frac{\sin 3x-\sin x}{\sin x}$

We know,

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

Therefore,

$\lim_{x\to 0}\frac{\sin 3x-\sin x}{\sin x}$

$=\lim_{x\to 0}\frac{2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}}{\sin x}$

$=\lim_{x\to 0}\frac{2\cos\frac{4x}{2}\sin\frac{2x}{2}}{\sin x}$

$=2\times\lim_{x\to 0}\frac{\cos 2x\sin x}{\sin x}$

$=2\times\lim_{x\to 0}\cos 2x$

$=2\times\cos(2\times 0)$

$=2\times\cos 0$

{∵ cos 0 = 1}

= 2 × 1

= 2

Hence, the value of $\lim_{x\to 0}\frac{\sin 3x-\sin x}{\sin x}=2$

23. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}$$

Answer

To find: $\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}$

We know,

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

Therefore,

$$\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}$$

$$=\lim_{x\to 0}\frac{2\cos\frac{5x+3x}{2}\sin\frac{5x-3x}{2}}{\sin x}$$

$$=\lim_{x\to 0}\frac{2\cos\frac{8x}{2}\sin\frac{2x}{2}}{\sin x}$$

$$=2\times\lim_{x\to 0}\frac{\cos 4x\sin x}{\sin x}$$

$$=2\times\lim_{x\to 0}\cos 4x$$

=2 × cos(4 × 0

=2 × cos 0

{∵ cos 0 = 1}

= 2 × 1

= 2

Hence, the value of $\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}=2$

24. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\cos 3x-\cos 5x}{x^2}$$

Answer

To find: $\lim_{x\to 0}\frac{\cos 3x-\cos 5x}{x^2}$

We know,

$$\cos A-\cos B=2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$$

Therefore,

$$\lim_{x\to 0}\frac{\cos 3x-\cos 5x}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin\frac{3x+5x}{2}\sin\frac{5x-3x}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin\frac{8x}{2}\sin\frac{2x}{2}}{x^2}$$

$$=2\times\lim_{x\to 0}\frac{\sin 4x}{x}\times\frac{\sin x}{x}$$

Multiplying and dividing by 10:

$$=2\times 4\times\lim_{x\to 0}\frac{\sin 4x}{4x}\times\frac{\sin x}{x}$$

As,

$X \to 0 \Rightarrow 4x \to 0$

$$\lim_{x\to0} A(x) \times B(x) = \lim_{x\to0} A(x) \times \lim_{x\to0} B(x)$$

$$= 8 \times \lim_{4x\to0} \frac{\sin 4x}{4x} \times \lim_{x\to0} \frac{\sin x}{x}$$

Put 4x = k;

$$= 8 \times \lim_{k\to0} \frac{\sin k}{k} \times \lim_{x\to0} \frac{\sin x}{x}$$

Formula used:

$$\lim_{y\to0} \frac{\sin y}{y} = 1$$

Therefore,

$$\lim_{x\to0} \frac{\cos 3x - \cos 7x}{x^2}$$

$$= 8 \times \lim_{k\to0} \frac{\sin k}{k} \times \lim_{x\to0} \frac{\sin x}{x}$$

= 8 × 1

= 8

Hence, the value of $\lim_{x\to0} \frac{\cos 3x - \cos 5x}{x^2} = 8$

25. Question

Evaluate the following limits:

$$\lim_{x\to0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$$

Answer

To find: $\lim_{x\to0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$

$$\lim_{x\to0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$$

Dividing numerator and denominator by x:

$$= \lim_{x\to0} \frac{\frac{\tan 3x - 2x}{x}}{\frac{3x - \sin^2 x}{x}}$$

$$= \lim_{x\to0} \frac{\frac{\tan 3x}{x} - \frac{2x}{x}}{\frac{3x}{x} - \frac{\sin^2 x}{x}}$$

$$= \lim_{x\to0} \frac{\frac{\tan 3x}{x} - 2}{3 - \frac{\sin^2 x}{x}}$$

We know,

$$\lim_{x\to 0}\frac{A(x)-B(x)}{C(x)+D(x)}=\frac{\lim_{x\to 0}A(x)-\lim_{x\to 0}B(x)}{\lim_{x\to 0}C(x)-\lim_{x\to 0}D(x)}$$

Therefore,

$$=\frac{\lim_{x\to 0}\frac{\tan 3x}{x}-\lim_{x\to 0}2}{\lim_{x\to 0}3+\lim_{x\to 0}\frac{\sin^2 x}{x}}$$

$$=\frac{\lim_{x\to 0}\left(\frac{\tan 3x}{3x}\right)\times 3-\lim_{x\to 0}2}{\lim_{x\to 0}3+\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)\times x}$$

$$=\frac{3\lim_{x\to 0}\frac{\tan 3x}{3x}-2}{3+\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times x}$$

As, $x\to 0\Rightarrow 3x\to 0$

$$=\frac{3\lim_{3x\to 0}\frac{\tan 3x}{3x}-2}{3+\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times x}$$

Put 3x = y:

$$=\frac{3\lim_{y\to 0}\frac{\tan y}{y}-2}{3+\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times x}$$

Formula used:

$$\lim_{x\to 0}\frac{\sin x}{x}=1\ \&\ \lim_{x\to 0}\frac{\tan x}{x}=1$$

Therefore,

$$\lim_{x\to 0}\frac{\tan 3x-2x}{3x-\sin^2 x}$$

$$=\frac{3\lim_{y\to 0}\frac{\tan y}{y}-2}{3+\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times x}$$

$$=\frac{3-2}{3+\lim_{x\to 0}x}$$

$$=\frac{3-2}{3+0}$$

$$=\frac{1}{3}$$

Hence , the value of $\lim_{x\to 0}\frac{\tan 3x-2x}{3x-\sin^2 x}=\frac{1}{3}$

26. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$$

Answer

To find: $\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$

We know,

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

Therefore,

$$\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$$

$$=\lim_{x\to 0}\frac{2\cos\frac{2+x+2-x}{2}\sin\frac{2+x-(2-x)}{2}}{x}$$

$$=\lim_{x\to 0}\frac{2\cos\frac{4}{2}\sin\frac{2+x-2+x}{2}}{x}$$

$$=\lim_{x\to 0}\frac{2\cos\frac{4}{2}\sin\frac{2x}{2}}{x}$$

$$=\lim_{x\to 0}\frac{2\cos 2\sin x}{x}$$

$$=2\cos 2\times\lim_{x\to 0}\frac{\sin x}{x}$$

Formula used:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Therefore,

$$\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$$

$$=2\cos 2\times\lim_{x\to 0}\frac{\sin x}{x}$$

= 2 cos 2 × 1

= 2 cos 2

Hence, the value of $\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}=2\cos 2$

27. Question

Evaluate the following limits:

$$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$$

Answer

To find: $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

We know,

$(a + b)^2 = a^2 + b^2 + 2ab$

Therefore,

$$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$$

$$=\lim_{h\to 0}\frac{(a^2+h^2+2ah)\sin(a+h)-a^2\sin a}{h}$$

$$=\lim_{h\to 0}\frac{a^2\sin(a+h)+h^2\sin(a+h)+2ah\sin(a+h)-a^2\sin a}{h}$$

$$=\lim_{h\to 0}\frac{a^2\{\sin(a+h)-\sin a\}}{h}+\frac{h^2\sin(a+h)}{h}+\frac{2ah\sin(a+h)}{h}$$

Now,

$$\lim_{x\to 0}A(x)+B(x)+C(x)=\lim_{x\to 0}A(x)+\lim_{x\to 0}B(x)+\lim_{x\to 0}C(x)\ \&$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

We get,

$$=\lim_{h\to 0}\frac{a^2\left\{2\cos\frac{a+h+a}{2}\sin\frac{a+h-a}{2}\right\}}{h}+\lim_{h\to 0}\frac{h^2\sin(a+h)}{h}+\lim_{h\to 0}\frac{2ah\sin(a+h)}{h}$$

$$=\lim_{h\to 0}\frac{a^2\left\{2\cos\frac{2a+h}{2}\sin\frac{h}{2}\right\}}{h}+\lim_{h\to 0}h\sin(a+h)+\lim_{h\to 0}2a\sin(a+h)$$

$$=\lim_{h\to 0}2a^2\cos\left(\frac{2a+h}{2}\right)\times\frac{\sin\frac{h}{2}}{2\times\frac{h}{2}}+0\times\sin(a+0)+2a\sin(a+0)$$

$$=\lim_{h\to 0}a^2\cos\left(\frac{2a+h}{2}\right)\times\frac{\sin\frac{h}{2}}{\frac{h}{2}}+0+2a\sin a$$

Formula used:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Therefore,

$$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$$

$$=a^2\cos\left(\frac{2a+0}{2}\right)\times 1+2a\sin a$$

$$=a^2\cos\left(\frac{2a}{2}\right)+2a\sin a$$

$$=a^2\cos a+2a\sin a$$

Hence, the value of $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}=a^2\cos a+2a\sin a$

28. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin 3x-3\sin x}$$

Answer

To find: $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin 3x-3\sin x}$

We know,

$\tan x=\frac{\sin x}{\cos x}$ & $\sin 3x=3\sin x-4\sin^3 x$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin 3x-3\sin x}$$

$$=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{3\sin x-4\sin^3 x-3\sin x}$$

$$=\lim_{x\to 0}\frac{\sin x\left(\frac{1}{\cos x}-1\right)}{-4\sin^3 x}$$

$$=-\frac{1}{4}\times\lim_{x\to 0}\frac{\frac{1}{\cos x}-1}{\sin^2 x}$$

$$=-\frac{1}{4}\times\lim_{x\to 0}\frac{\frac{1-\cos x}{\cos x}}{1-\cos^2 x}$$

{∵ $\sin^2 x = 1 - \cos^2 x$}

$$=-\frac{1}{4}\times\lim_{x\to 0}\frac{(1-\cos x)}{\cos x\,(1-\cos x)\,(1+\cos x)}$$

{∵ $a^2 - b^2 = (a - b)(a+b)$}

$$=-\frac{1}{4}\times\lim_{x\to 0}\frac{1}{\cos x\,(1+\cos x)}$$

$$=-\frac{1}{4}\times\frac{1}{\cos 0\,(1+\cos 0)}$$

{∵ $\cos 0 = 1$}

$$=-\frac{1}{4}\times\frac{1}{(1+1)}$$

$$=-\frac{1}{4}\times\frac{1}{2}$$

$$=-\frac{1}{8}$$

Hence, the value of $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin 3x-3\sin x}=-\frac{1}{8}$

29. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sec 5x-\sec 3x}{\sec 3x-\sec x}$$

Answer

To find: $\lim_{x\to 0}\frac{\sec 5x-\sec 3x}{\sec 3x-\sec x}$

We know,

$$\sec x=\frac{1}{\cos x}$$

$$\lim_{x\to 0}\frac{\sec 5x-\sec 3x}{\sec 3x-\sec x}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\cos 5x}-\frac{1}{\cos 3x}}{\frac{1}{\cos 3x}-\frac{1}{\cos x}}$$

$$=\lim_{x\to 0}\frac{\frac{\cos 3x-\cos 5x}{\cos 5x\cos 3x}}{\frac{\cos x-\cos 3x}{\cos 3x\cos x}}$$

$$=\lim_{x\to 0}\frac{\cos 3x-\cos 5x}{\cos x-\cos 3x}\times\frac{\cos 5x\cos 3x}{\cos 3x\cos x}$$

$$=\lim_{x\to 0}\frac{\cos 3x-\cos 5x}{\cos x-\cos 3x}\times\frac{\cos 5x}{\cos x}$$

We know,

$$\cos A-\cos B=2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$$

$$=\lim_{x\to 0}\frac{2\sin\frac{3x+5x}{2}\sin\frac{5x-3x}{2}}{2\sin\frac{x+3x}{2}\sin\frac{3x-x}{2}}\times\frac{\cos 5x}{\cos x}$$

$$=\lim_{x\to 0}\frac{\sin\frac{8x}{2}\sin\frac{2x}{2}}{\sin\frac{4x}{2}\sin\frac{2x}{2}}\times\frac{\cos 5x}{\cos x}$$

$$=\lim_{x\to 0}\frac{\sin 4x\sin x}{\sin 2x\sin x}\times\frac{\cos 5x}{\cos x}$$

$$=\lim_{x\to 0}\frac{\sin 4x\cos 5x}{\sin 2x\cos x}$$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin 4x}{4x}\right)\times 4x\times\cos 5x}{\left(\frac{\sin 2x}{2x}\right)\times 2x\times\cos x}$$

$$=2\times\lim_{x\to 0}\frac{\left(\frac{\sin 4x}{4x}\right)\times\cos 5x}{\left(\frac{\sin 2x}{2x}\right)\times\cos x}$$

We know,

$$\lim_{x\to 0}\frac{A(x)\times B(x)}{C(x)\times D(x)}=\frac{\lim_{x\to 0}A(x)\times\lim_{x\to 0}B(x)}{\lim_{x\to 0}C(x)\times\lim_{x\to 0}D(x)}$$

Therefore,

$$= 2 \times \frac{\lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right) \times \lim_{x\to 0} \cos 5x}{\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right) \times \lim_{x\to 0} \cos x}$$

As, x → 0 ⇒ 3x → 0 & 4x → 0

$$= 2 \times \frac{\lim_{4x\to 0}\left(\frac{\sin 4x}{4x}\right) \times \lim_{x\to 0} \cos 5x}{\lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right) \times \lim_{x\to 0} \cos x}$$

Put 2x = y & 4x = t:

$$= 2 \times \frac{\lim_{t\to 0}\left(\frac{\sin t}{t}\right) \times \lim_{x\to 0} \cos 5x}{\lim_{y\to 0}\left(\frac{\sin y}{y}\right) \times \lim_{x\to 0} \cos x}$$

Formula used:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Therefore,

$$\lim_{x\to 0}\frac{1 - \cos 2x}{\cos 2x - \cos 8x}$$

$$= 2 \times \frac{\lim_{t\to 0}\left(\frac{\sin t}{t}\right) \times \lim_{x\to 0} \cos 5x}{\lim_{y\to 0}\left(\frac{\sin y}{y}\right) \times \lim_{x\to 0} \cos x}$$

$$= 2 \times \frac{1 \times \cos(5 \times 0)}{1 \times \cos 0}$$

$$= 2 \times \frac{1 \times \cos 0}{1 \times \cos 0}$$

= 2

Hence, the value of $\lim_{x\to 0}\frac{\sec 5x - \sec 3x}{\sec 3x - \sec x} = 2$

30. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{1 - \cos 2x}{\cos 2x - \cos 8x}$$

Answer

To find: $\lim_{x\to 0}\frac{1 - \cos 2x}{\cos 2x - \cos 8x}$

We know,

$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$$

$\cos 2x = 1 - 2\sin^2 x$

⇒ $2\sin^2 x = 1 - \cos 2x$

$$\lim_{x\to 0}\frac{1-\cos 2x}{\cos 2x-\cos 8x}$$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{2\sin\frac{2x+8x}{2}\sin\frac{8x-2x}{2}}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin\frac{10x}{2}\sin\frac{6x}{2}}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin 5x\sin 3x}$$

Dividing numerator and denominator by x^2:

$$=\lim_{x\to 0}\frac{\frac{\sin^2 x}{x^2}}{\frac{\sin 5x\sin 3x}{x^2}}$$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)^2}{\frac{\sin 5x}{x}\times\frac{\sin 3x}{x}}$$

We know,

$$\lim_{x\to 0}\frac{A(x)}{C(x)\times D(x)}=\frac{\lim_{x\to 0}A(x)}{\lim_{x\to 0}C(x)\times\lim_{x\to 0}D(x)}$$

Therefore,

$$=\frac{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}{\lim_{x\to 0}\frac{\sin 5x}{x}\times\lim_{x\to 0}\frac{\sin 3x}{x}}$$

$$=\frac{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}{\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)\times 5\times\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\times 3}$$

$$=\frac{1}{15}\times\frac{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}{\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)\times\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)}$$

$\text{As}, x\to 0\Rightarrow 3x\to 0\ \&\ 5x\to 0$

$$=\frac{1}{15}\times\frac{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}{\lim_{5x\to 0}\left(\frac{\sin 5x}{5x}\right)\times\lim_{3x\to 0}\left(\frac{\sin 3x}{3x}\right)}$$

Put $3x = y$ & $5x = t$:

$$=\frac{1}{15}\times\frac{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}{\lim_{t\to 0}\left(\frac{\sin t}{t}\right)\times\lim_{y\to 0}\left(\frac{\sin y}{y}\right)}$$

Formula used:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Therefore,

$$\lim_{x\to 0}\frac{\sec 5x-\sec 3x}{\sec 3x-\sec x}$$

$$=\frac{1}{15}\times\frac{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}{\lim_{t\to 0}\left(\frac{\sin t}{t}\right)\times\lim_{y\to 0}\left(\frac{\sin y}{y}\right)}$$

$$=\frac{1}{15}\times\frac{(1)^2}{1}$$

$$=\frac{1}{15}$$

Hence, the value of $\lim_{x\to 0}\frac{1-\cos 2x}{\cos 2x-\cos 8x}=\frac{1}{15}$

31. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{1-\cos 2x+\tan^2 x}{x\sin x}$$

Answer

$$\lim_{x\to 0}\frac{1-\cos 2x+\tan^2 x}{x\sin x}$$

Now, 1 - cos2x = 2 $\sin^2$x

$$=\lim_{x\to 0}\frac{2\sin^2 x+\tan^2 x}{x\sin x}$$

$$=\frac{2\lim_{x\to 0}\sin^2 x+\lim_{x\to 0}\tan^2 x}{\lim_{x\to 0}x\sin x}$$

$$=\frac{2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2+\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2\times x^2}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times x^2}$$

$$=\frac{(2\times 1\times x^2)+(1\times x^2)}{(1\times x^2)}$$

Since, $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{\tan x}{x}=1$

$$\Rightarrow\lim_{x\to 0}\frac{1-\cos 2x+\tan^2 x}{x\sin x}=\frac{3x^2}{x^2}$$

$$\Rightarrow\lim_{x\to 0}\frac{1-\cos 2x+\tan^2 x}{x\sin x}=3$$

Hence, $\lim_{x\to 0}\frac{1-\cos 2x+\tan^2 x}{x\sin x}=3$

32. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin(a+x)+\sin(a-x)-2\sin a}{x\sin x}$$

Answer

$$\lim_{x\to 0}\frac{\sin(a+x)+\sin(a-x)-2\sin a}{x\sin x}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{a+x+a-x}{2}\right)\cos\left(\frac{a+x-a+x}{2}\right)-2\sin a}{x\sin x}$$

$$= \lim_{x\to 0}\frac{2\sin a(\cos x-1)}{x\sin x}$$

$$= -2\sin a\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}$$

$$= -2\sin a\lim_{x\to 0}\frac{2\sin\frac{x}{2}}{x\left(\cos\frac{x}{2}\right)}$$

$$= -2\sin a\lim_{x\to 0}\frac{\tan\frac{x}{2}}{\frac{x}{2}}\times\frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0}\frac{\sin(a+x)+\sin(a-x)-2\sin a}{x\sin x} = -2\sin a\times 1\times\frac{1}{2}$$

Since, $\lim_{x\to 0}\frac{\tan x}{x} = 1$

$$\Rightarrow \lim_{x\to 0}\frac{\sin(a+x)+\sin(a-x)-2\sin a}{x\sin x} = -\sin a$$

33. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{x^2-\tan 2x}{\tan x}$$

Answer

$$\lim_{x\to 0}\frac{x^2-\tan 2x}{\tan x}$$

$$= \lim_{x\to 0}\frac{\frac{x^2}{2x}-\frac{\tan 2x}{2x}}{\frac{\tan x}{x}x}$$

$$= \lim_{x\to 0}\frac{\left[\frac{x^2}{2x}-\frac{\tan 2x}{2x}\right]2x}{\frac{\tan x}{x}x}$$

$$= \lim_{x\to 0}\frac{\left[\frac{x^2}{2x}-\frac{\tan 2x}{2x}\right]2}{\frac{\tan x}{x}}$$

$$\Rightarrow \lim_{x\to 0}\frac{x^2-\tan 2x}{\tan x} = 2\left[\frac{0-1}{1}\right]$$

$$\Rightarrow \lim_{x\to 0}\frac{x^2-\tan 2x}{\tan x} = -2$$

34. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$$

Answer

$\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$

Rationalize the numerator, we get $\Rightarrow \lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \lim_{x \to 0} \frac{\sqrt{2} + \sqrt{1 + \cos x}}{\sin^2 x} \times \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sqrt{2} - \sqrt{1 + \cos x}}$

$$= \lim_{x \to 0} \frac{2 - 1 - \cos x}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{1 - \cos^2 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{(1 + \cos x)(1 - \cos x)}$$

$$= \lim_{x \to 0} \frac{1}{(1 + \cos x)}$$

$$= \frac{1}{1 + \cos 0}$$

$$\Rightarrow \lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \frac{1}{1 + 1}$$

Hence, $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \frac{1}{2}$

35. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{x \tan x}{1 - \cos x}$$

Answer

$\lim_{x \to 0} \frac{x \tan x}{1 - \cos x}$

$$\Rightarrow \lim_{x \to 0} \frac{x \tan x}{1 - \cos x} = \lim_{x \to 0} \frac{x \frac{\sin x}{\cos x}}{1 - \cos x}$$

$$= \lim_{x \to 0} \frac{x \sin x}{\cos x (1 - \cos x)}$$

$$= \lim_{x \to 0} \frac{x \left(2 \sin \frac{x}{2} \cos \frac{x}{2}\right)}{\cos x \left(2 \sin^2 \frac{x}{2}\right)}$$

$$= \lim_{x\to 0}\frac{x\cos\frac{x}{2}}{\cos x\left(\sin\frac{x}{2}\right)}$$

$$= \lim_{x\to 0}\frac{1}{\frac{\cos x\left(\frac{\tan x}{2}\right)}{x}}$$

$$= \lim_{x\to 0}\frac{1}{\cos x}\times\frac{1}{\lim_{x\to 0}\frac{\frac{\tan x}{2}}{\frac{x}{2}}\times\frac{1}{2}}$$

$$\Rightarrow \lim_{x\to 0}\frac{x\tan x}{1-\cos x} = 1\times 2\times 1$$

Hence, $\lim_{x\to 0}\frac{x\tan x}{1-\cos x} = 2$

36. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{x^2+1-\cos x}{x\sin x}$$

Answer

$\lim_{x\to 0}\frac{x^2+1-\cos x}{x\sin x}$

$$\Rightarrow \lim_{x\to 0}\frac{x^2+1-\cos x}{x\sin x} = \lim_{x\to 0}\frac{x^2+2\sin^2\frac{x}{2}}{x\sin x}$$

$$= \lim_{x\to 0}\frac{x^2\left[1+2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\right]}{x\sin x}$$

$$= \lim_{x\to 0}\frac{\left[1+2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\times\frac{1}{4}\right]}{\frac{\sin x}{x}}$$

$$= \frac{\left[1+2\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\times\frac{1}{4}\right]}{\lim_{x\to 0}\frac{\sin x}{x}}$$

$$= \frac{1+2\times 1\times\frac{1}{4}}{1} = 1+\frac{1}{2}$$

Hence, $\lim_{x\to 0}\frac{x^2+1-\cos x}{x\sin x} = \frac{1+2\times 1\times\frac{1}{4}}{1} = \frac{3}{2}$

37. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin 2x(\cos 3x - \cos x)}{x^3}$$

Answer

$$\lim_{x\to 0}\frac{\sin 2x(\cos 3x - \cos x)}{x^3}$$

Since, $\cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$$= \lim_{x\to 0}\frac{\sin 2x\left(-2\sin\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{\sin 2x(-2\sin 2x \sin x)}{x^3}$$

$$= \frac{-2\lim\limits_{x\to 0}\sin 2x \times \lim\limits_{x\to 0}\sin 2x \times \lim\limits_{x\to 0}\sin x}{x^3}$$

$$= -2\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\times 2\right)\times\left(2\lim_{x\to 0}\frac{\sin 2x}{2x}\right)\times\left(\lim_{x\to 0}\frac{\sin x}{x}\right)$$

= -2(1 × 2) × 2 × 1

Hence, $\lim_{x\to 0}\frac{\sin 2x(\cos 3x - \cos x)}{x^3} = -8$

38. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{2\sin x^0 - \sin 2x^0}{x^3}$$

Answer

$$\lim_{x\to 0}\frac{2\sin x^0 - \sin 2x^0}{x^3}$$

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x^0 - \sin 2x^0}{x^3} = \lim_{x\to 0}\frac{2\sin\frac{\pi x}{180} - \sin\frac{2x\pi}{180}}{x^3}$$

$$= \lim_{x\to 0}\frac{2\sin\frac{\pi x}{180} - 2\sin\frac{x\pi}{180}\cos\frac{\pi x}{180}}{x^3}$$

$$= \lim_{x\to 0}\frac{2\sin\frac{\pi x}{180}\left(1-\cos\frac{\pi x}{180}\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{2\sin\frac{\pi x}{180}\left(2\sin^2\frac{x\pi}{360}\right)}{x^3}$$

$$= 4\left(\lim_{x\to 0}\frac{\sin\frac{x\pi}{180}}{x}\right)\times\left(\lim_{x\to 0}\frac{\sin\frac{x\pi}{360}}{x}\right)\times\left(\lim_{x\to 0}\frac{\sin\frac{x\pi}{360}}{x}\right)$$

$$= 4\left(\lim_{x\to 0}\frac{\sin\frac{x\pi}{180}}{x\frac{\pi}{180}}\times\frac{\pi}{180}\right)\times\left(\lim_{x\to 0}\frac{\sin\frac{x\pi}{360}}{x\frac{\pi}{360}}\times\frac{\pi}{360}\right)\times\left(\lim_{x\to 0}\frac{\sin\frac{x\pi}{360}}{x\frac{\pi}{360}}\times\frac{\pi}{360}\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x^0 - \sin 2x^0}{x^3} = 4\times\frac{\pi}{180}\times\frac{\pi}{360}\times\frac{\pi}{360}$$

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x^0 - \sin 2x^0}{x^3} = \left(\frac{\pi}{180}\right)^3$$

Hence, $\lim_{x\to 0}\frac{2\sin x^0 - \sin 2x^0}{x^3} = \left(\frac{\pi}{180}\right)^3$

39. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}$$

Answer

$\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}$

$$\Rightarrow \lim_{x\to 0}\frac{x^3\cot x}{1-\cos x} = \lim_{x\to 0}\frac{x^3\frac{1}{\tan x}}{1-\cos x}$$

$$= \lim_{x\to 0}\frac{x^3}{\tan x\,(1-\cos x)}$$

$$= \lim_{x\to 0}\frac{x^3}{\tan x\,(2\sin^2\frac{x}{2})}$$

$$= \lim_{x\to 0}\frac{1}{\frac{\tan x}{x}\times\frac{2\sin^2\frac{x}{2}}{x^2}}$$

$$= \frac{1}{\lim_{x\to 0}\frac{\tan x}{x}\left[\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2\times\frac{1}{4}}$$

Since, $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\frac{\tan x}{x} = 1$

$$\Rightarrow \lim_{x\to 0}\frac{x^3\cot x}{1-\cos x} = \frac{1}{1\times 2\times\frac{1}{4}}$$

$$\Rightarrow \lim_{x\to 0}\frac{x^3\cot x}{1-\cos x} = 2$$

Hence, $\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x} = 2$

40. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{x\tan x}{1-\cos 2x}$$

Answer

$\lim_{x\to 0}\frac{x\tan x}{1-\cos 2x}$

Since, 1 - cos 2x = 2sin^{2}x

$$\Rightarrow \lim_{x\to 0}\frac{x\tan x}{1-\cos 2x} = \lim_{x\to 0}\frac{x\tan x}{2\sin^2 x}$$

$$= \lim_{x\to 0}\frac{\frac{\tan x}{x}}{\frac{2\sin^2 x}{x^2}}$$

$$= \frac{\lim_{x\to 0}\frac{\tan x}{x}}{2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}$$

Since, $\lim_{x\to 0}\frac{\tan x}{x} = 1$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$$\Rightarrow \lim_{x\to 0}\frac{x\tan x}{1-\cos 2x} = \frac{1}{2\times 1}$$

Hence, $\lim_{x\to 0}\frac{x\tan x}{1-\cos 2x} = \frac{1}{2}$

41. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin(3+x)-\sin(3-x)}{x}$$

Answer

$\lim_{x\to 0}\frac{\sin(3+x)-\sin(3-x)}{x}$

$$= \lim_{x\to 0}\frac{2\cos\left(\frac{3+x+3-x}{2}\right)\sin\left(\frac{3+x-3+x}{2}\right)}{x}$$

$$= 2\lim_{x\to 0}\frac{\cos\left(\frac{3+x+3-x}{2}\right)\sin\left(\frac{3+x-3+x}{2}\right)}{x}$$

$$= 2\lim_{x\to 0}\frac{\cos 3.\sin x}{x}$$

$$= 2\cos 3.\lim_{x\to 0}\frac{\sin x}{x}$$

= 2 cos 3

Hence, $\lim_{x\to 0}\frac{\sin(3+x)-\sin(3-x)}{x} = 2\cos 3$

42. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$$

Answer

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$$

We know that, cos2x = 1 – 2sin^{2}x

Therefore,

$$\Rightarrow \lim_{x \to 0} \frac{1 - 2\sin^2 x - 1}{\cos x - 1}$$

$$= \lim_{x \to 0} \frac{(-2\sin^2 x)}{\cos x - 1}$$

$$= \lim_{x \to 0} \left(-\frac{2(1 - \cos^2 x))}{\cos x - 1}\right)$$

[cos^{2}x – 1 = (cosx + 1)(cosx – 1)]

$$= \lim_{x \to 0} 2(1 + \cos x)$$

= 2(1 + 0)

= 2

Hence, $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1} = 2$

43. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2}$$

Answer

$$\lim_{x \to 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2}$$

$$\Rightarrow \lim_{x \to 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2} = \lim_{x \to 0} \frac{3\sin^2 x}{3x^2} - \lim_{x \to 0} \frac{2\sin x^2}{3x^2}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 - \frac{2}{3} \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2$$

Since, $\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$

$$= 1 - \frac{2}{3}$$

$$\Rightarrow \lim_{x \to 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2} = \frac{1}{3}$$

Hence, $\lim_{x \to 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2} = \frac{1}{3}$

44. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$$

Answer

$\lim_{x \to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$

$$= \lim_{x \to 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x} \times \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}$$

$$= \lim_{x \to 0} \frac{(1 + \sin x) - (1 - \sin x)}{x(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})}$$

$$= \lim_{x \to 0} \frac{2 \sin x}{x(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})}$$

$$= 2. \lim_{x \to 0} \frac{\sin x}{x} \frac{1}{\lim_{x \to 0}(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})}$$

$$\Rightarrow \lim_{x \to 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x} = 2. \times 1 \times \frac{1}{2}$$

Hence, $\lim_{x \to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x} = 1$

45. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{1 - \cos 4x}{x^2}$$

Answer

$\lim_{x \to 0} \frac{1-\cos 4x}{x^2}$

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos 4x}{x^2} = \lim_{x \to 0} \frac{2 \sin^2 2x}{x^2}$$

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos 4x}{x^2} = 2 \lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)^2$$

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos 4x}{x^2} = 2 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)^2 \times 2^2$$

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos 4x}{x^2} = 2 \times 1 \times 4$$

Hence, $\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = 8$

46. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{x \cos x + \sin x}{x^2 + \tan x}$$

Answer

$\lim_{x\to 0}\frac{x\cos x+\sin x}{x^2+\tan x}$

$$\Rightarrow \lim_{x\to 0}\frac{x\cos x + \sin x}{x^2 + \tan x} = \lim_{x\to 0}\frac{\cos x + \frac{\sin x}{x}}{x + \frac{\tan x}{x}}$$

$$\Rightarrow \lim_{x\to 0}\frac{x\cos x + \sin x}{x^2 + \tan x} = \frac{\lim_{x\to 0}\cos x + \lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0}x + \lim_{x\to 0}\frac{\tan x}{x}}$$

$$\Rightarrow \lim_{x\to 0}\frac{x\cos x + \sin x}{x^2 + \tan x} = \frac{1 + 1}{0 + 1}$$

Hence, $\lim_{x\to 0}\frac{x\cos x+\sin x}{x^2+\tan x} = 2$

47. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{1-\cos 2x}{3\tan^2 x}$$

Answer

$\lim_{x\to 0}\frac{1-\cos 2x}{3\tan^2 x}$

Since, 1 - cos2x = 2sin^2 x

$$\Rightarrow \lim_{x\to 0}\frac{1 - \cos 2x}{3\tan^2 x} = \lim_{x\to 0}\frac{2\sin^2 x}{3\tan^2 x}$$

$$= \frac{2}{3}\lim_{x\to 0}\frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{2}{3}\lim_{x\to 0}\cos^2 x$$

$$= \frac{2}{3}\lim_{x\to 0}\cos^2 0$$

$$\Rightarrow \lim_{x\to 0}\frac{1 - \cos 2x}{3\tan^2 x} = \frac{2}{3}$$

Hence, $\lim_{x\to 0}\frac{1-\cos 2x}{3\tan^2 x} = \frac{2}{3}$

48. Question

Evaluate the following limits:

$$\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta}$$

Answer

$\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta}$

$$\Rightarrow \lim_{\theta\to 0}\frac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta\to 0}\frac{2\sin^2 2\theta}{2\sin^2 3\theta}$$

$$= \lim_{\theta \to 0} \frac{(\sin 2\theta)^2}{(\sin 3\theta)^2}$$

$$= \frac{\lim_{\theta \to 0} \left(\frac{\sin 2\theta}{2\theta}\right)^2 \times 4\theta^2}{\lim_{\theta \to 0} \left(\frac{\sin 3\theta}{3\theta}\right)^2 \times 9\theta^2}$$

$$= \frac{1^2 \times 4\theta^2}{1 \times 9\theta^2}$$

$$= \frac{4}{9}$$

Hence, $\lim_{\theta \to 0} \frac{1-\cos 4\theta}{1-\cos 6\theta} = \frac{4}{9}$

49. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{ax + x\cos x}{b\sin x}$$

Answer

$\lim_{x \to 0} \frac{ax + x\cos x}{b\sin x}$

$$\Rightarrow \lim_{x \to 0} \frac{ax + x\cos x}{b\sin x} = \lim_{x \to 0} \frac{a + \cos x}{\frac{b\sin x}{x}}$$

$$\Rightarrow \lim_{x \to 0} \frac{ax + x\cos x}{b\sin x} = \frac{\lim_{x \to 0} (a + \cos x)}{\lim_{x \to 0} \frac{b\sin x}{x}}$$

$$\Rightarrow \lim_{x \to 0} \frac{ax + x\cos x}{b\sin x} = \frac{a + 1}{b}$$

Hence, $\lim_{x \to 0} \frac{ax + x\cos x}{b\sin x} = \frac{a+1}{b}$

50. Question

Evaluate the following limits:

$$\lim_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta}$$

Answer

$\lim_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta}$

$$\Rightarrow \lim_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta} = \frac{\lim_{\theta \to 0} \frac{\sin 4\theta}{4\theta} \times 4\theta}{\lim_{\theta \to 0} \frac{\tan 3\theta}{3\theta} \times 3\theta}$$

$$\Rightarrow \lim_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta} = \frac{1 \times 4\theta}{1 \times 3\theta}$$

$$\Rightarrow \lim_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta} = \frac{4}{3}$$

Hence, $\lim_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta} = \frac{4}{3}$

51. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$$

Answer

$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$

Since, sin 2x = 2 sinx .cos x

$$\Rightarrow \lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3} = \lim_{x \to 0} \frac{2\sin x - (2\sin x.\cos x)}{x^3}$$

$$= \lim_{x \to 0} \frac{2\sin x(1 - \cos x)}{x^3}$$

$$= \lim_{x \to 0} \frac{2\sin x(1 - \cos x)}{x^3} \times \frac{(1 + \cos x)}{(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{2\sin x(1 - \cos x)^2}{x^3((1 + \cos x))}$$

$$= \lim_{x \to 0} \frac{2\sin x(\sin^2 x)}{x^3((1 + \cos x))}$$

$$= \lim_{x \to 0} \frac{2\sin^3 x}{x^3((1 + \cos x))}$$

$$= 2\lim_{x \to 0} \frac{\sin^3 x}{x^3((1 + \cos x))}$$

$$= 2\lim_{x \to 0} \left(\frac{\sin}{x}\right)^3 \times \lim_{x \to 0} \frac{1}{(1 + \cos x)}$$

$$\Rightarrow \lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3} = 2 \times 1 \times \frac{1}{2}$$

Hence, $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3} = 1$

52. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{1 - \cos 5x}{1 - \cos 6x}$$

Answer

$\lim_{x \to 0} \frac{1 - \cos 5x}{1 - \cos 6x}$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{5x}{2}}{2\sin^2 \frac{3x}{2}}$$

$$= \frac{\lim_{x\to 0}\left(\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}\right)^2 \times \frac{25}{4}x^2}{\lim_{x\to 0}\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2 \times \frac{9}{4}x^2}$$

$$= \frac{2 \times \frac{25}{4}x^2}{2 \times 1 \times 9x^2}$$

$$\Rightarrow \lim_{x\to 0}\frac{1-\cos 5x}{1-\cos 6x} = \frac{25}{4 \times 9}$$

Hence, $\lim_{x\to 0}\frac{1-\cos 5x}{1-\cos 6x} = \frac{25}{36}$

53. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\operatorname{cosec} x - \cot x}{x}$$

Answer

Given, $\lim_{x\to 0}\frac{\operatorname{cose} x - \cot x}{x}$

$$\Rightarrow \lim_{x\to 0}\frac{\operatorname{cose} x - \cot x}{x} = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) \times \frac{1}{x}$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x}\left(\frac{1-\cos x}{x}\right)\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x}\left(\frac{2\sin^2\frac{x}{2}}{x}\right)\right)$$

$$= 2\lim_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}} \times x\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \times \frac{x}{4}\right)$$

$$= 2 \times \frac{1}{x} \times \frac{x}{4}$$

$$\Rightarrow \lim_{x\to 0}\frac{\operatorname{cose} x - \cot x}{x} = \frac{1}{2}$$

Hence, $\lim_{x\to 0}\frac{\operatorname{cose} x - \cot x}{x} = \frac{1}{2}$

54. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin 3x + 7x}{4x + \sin 2x}$$

Answer

Given, $\lim_{x\to 0}\frac{\sin 3x + 7x}{4x + \sin 2x}$

Now, divide by x

$$= \lim_{x\to 0}\frac{\frac{\sin 3x}{x}+\frac{7x}{x}}{\frac{4x}{x}+\frac{\sin 2x}{x}}$$

$$= \frac{\lim_{x\to 0}\frac{\sin 3x}{3x}\times 3 + 7}{4 + \lim_{x\to 0}\frac{\sin 2x}{2x}2}$$

$$= \frac{3 + 7}{4 + 2}$$

$$= \frac{10}{6}$$

Hence, $\lim_{x\to 0}\frac{\sin 3x + 7x}{4x + \sin 2x} = \frac{10}{6}$

55. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{5x + 4\sin 3x}{4\sin 2x + 7x}$$

Answer

Given, $\lim_{x\to 0}\frac{5x + 4\sin 3x}{4\sin 2x + 7x}$

$$\Rightarrow \lim_{x\to 0}\frac{5x + 4\sin 3x}{4\sin 2x + 7x} = \lim_{x\to 0}\frac{5 + \frac{4\sin 3x}{x}}{\frac{4\sin 2x}{x} + 7}$$

$$= \frac{5 + \left[\lim_{x\to 0}\frac{4\sin 3x}{3x}\times 3\right]}{\left[\lim_{x\to 0}\frac{4\sin 2x}{2x}\times 2\right] + 7}$$

$$= \frac{5 + 4\times 1\times 3}{4\times 2 + 7}$$

$$= \frac{5 + 12}{8 + 7}$$

$$= \frac{17}{15}$$

Hence, $\lim_{x\to 0}\frac{5x + 4\sin 3x}{4\sin 2x + 7x} = \frac{17}{15}$

56. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{3\sin x - \sin 3x}{x^3}$$

Answer

Given, $\lim_{x\to 0}\frac{3\sin x - \sin 3x}{x^3}$

Since, $\sin 3x = 3\sin x - 4\sin^3 x$

$$= \lim_{x\to 0}\frac{3\sin x-(3\sin x-4\sin^3 x)}{x^3}$$

$$= \lim_{x\to 0}\frac{4\sin^3 x}{x^3}$$

$$= 4\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

= 4 × 1

Hence, $\lim_{x\to 0}\frac{3\sin x-\sin 3x}{x^3} = 4$

57. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3}$$

Answer

$\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3}$

Put $\tan x = \frac{\sin x}{\cos x}$

$$= \lim_{x\to 0}\frac{\frac{\sin 2x}{\cos 2x}-\sin 2x}{x^3}$$

$$= \lim_{x\to 0}\frac{\sin 2x(\frac{1}{\cos 2x}-1)}{x^3}$$

$$= \lim_{x\to 0}\frac{\sin 2x(1-\cos 2x)}{x^3(\cos 2x)}$$

$$= \lim_{x\to 0}\frac{\sin 2x(2\sin^2 x)}{x^3(\cos 2x)}$$

$$= \frac{\lim_{x\to 0}\frac{\sin 2x}{x}\left(\lim_{x\to 0}\frac{2\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\cos 2x}$$

$$= \frac{\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\times 2\right)2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{\lim_{x\to 0}\cos 2x}$$

$$= \frac{(2\times 1)(2\times 1)}{1}$$

= 4

Hence, $\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3} = 4$

58. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$$

Answer

Given, $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$

Taking x as common, we get

$$\Rightarrow \lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx} = \lim_{x \to 0} \frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}}$$

$$= \frac{\lim_{x \to 0} \frac{\sin ax}{ax} \times a + b}{a + \lim_{x \to 0} \frac{\sin bx}{x} \times b}$$

$$= \frac{a + b}{a + b}$$

= 1

Hence, $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx} = 1$

59. Question

Evaluate the following limits:

$$\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$$

Answer

Given, $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

$$= \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x \to 0} \left(\frac{1 - \cos x}{\sin x}\right)$$

$$= \lim_{x \to 0} \left(\frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}\right)$$

$$= \lim_{x \to 0} \left(\frac{\tan \frac{x}{2}}{\frac{x}{2}}\right) \times \frac{x}{2}$$

$$= \lim_{x \to 0} (1) \times \frac{x}{2}$$

= 0

Hence, $\lim_{x \to 0} (\operatorname{cosec} x - \cot x) = 0$

60. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{(\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x}{\cos^2 \beta x - \cos^2 \alpha x}$$

Answer

Here, $\lim\limits_{x \to 0} \frac{(\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos^2 \beta x - \cos^2 \alpha x}$

$$= \lim_{x \to 0} \frac{(2\sin \frac{(\alpha+\beta+\alpha-\beta)}{2} x + \cos \frac{(\alpha+\beta-\alpha+\beta)}{2} x + 2\sin \alpha x \cos \alpha x}{(\cos \beta x - \cos\alpha x)(\cos\beta x + \cos\alpha x)}$$

$$= \lim_{x \to 0} \frac{\{2\sin\alpha x\cos\beta x + 2\sin\alpha x\cos\alpha x\}}{(\cos \beta x - \cos\alpha x)(\cos\beta x + \cos\alpha x)}$$

$$= \lim_{x \to 0} \frac{2\sin \alpha x(\cos\beta x + \cos\alpha x)}{(\cos \beta x - \cos\alpha x)(\cos\beta x + \cos\alpha x)}$$

$$= \lim_{x \to 0} \frac{2\sin \alpha x}{(\cos \beta x - \cos\alpha x)}$$

$$= \lim_{x \to 0} \frac{2\sin \alpha x}{(1 - 2\sin^2\left(\frac{\beta x}{2}\right) - 2\sin^2\left(\frac{\alpha x}{2}\right)}$$

$$= \lim_{x \to 0} \frac{2\sin \alpha x}{(2\sin^2\left(\frac{\alpha x}{2}\right) - 2\sin^2\left(\frac{\beta x}{2}\right)}$$

$$\Rightarrow \lim_{x \to 0} \frac{(\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos^2 \beta x - \cos^2 \alpha x} = \frac{2\alpha}{\alpha^2 - \beta^2}$$

Hence, $\lim\limits_{x \to 0} \frac{(\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos^2 \beta x - \cos^2 \alpha x} = \frac{2\alpha}{\alpha^2-\beta^2}$

61. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$$

Answer

$\lim\limits_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

Explanation: Here, $\lim\limits_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

$$= \lim_{x \to 0} \frac{1 - 2\sin^2\left(\frac{ax}{2}\right) - 1 + 2\sin^2\left(\frac{bx}{2}\right)}{1 - 2\sin^2\left(\frac{cx}{2}\right) - 1}$$

$$= \lim_{x \to 0} \frac{-2\sin^2\left(\frac{ax}{2}\right) + 2\sin^2\left(\frac{bx}{2}\right)}{-2\sin^2\left(\frac{cx}{2}\right)}$$

$$= \lim_{x \to 0} \frac{-\sin^2\left(\frac{ax}{2}\right)4a^2x^2 + \sin^2\left(\frac{bx}{2}\right)4b^2x^2}{-\sin^2\left(\frac{cx}{2}\right)4c^2x^2}$$

$$= \frac{-a^2 + b^2}{-c^2}$$

$$= \frac{b^2 - a^2}{c^2}$$

Hence, $\lim_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-1} = \frac{b^2-a^2}{c^2}$

62. Question

Evaluate the following limits:

$$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$$

Answer

Given, $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

Explanation: $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

$$= \lim_{h\to 0}\frac{(a\ +\ h)^2(\sin a\cos h\ +\ \cos a\sin h)-a^2\sin a}{h}$$

$$= \lim_{h\to 0}\frac{(a\ +\ h)^2(\sin a\cos h)-a^2\sin a\ +\ (a\ +\ h)^2\cos a\sin h}{h}$$

$$= \lim_{h\to 0}\frac{a^2\sin a(\cos h-1)\ +\ 2ah\sin a\cos h\ +\ h\text{^}2\sin a\cos h\ +\ (a\ +\ h)\text{^}2\cos a\sin h}{h}$$

$$= \lim_{h\to 0}\frac{a^2\sin a(\cos h-1)}{h}\ +\ \lim_{h\to 0}\frac{2ah\sin a\cos h}{h}\ +\ \lim_{h\to 0}\frac{h^2\sin a\cos h}{h}\ +\ \lim_{h\to 0}\frac{(a\ +\ h)^2\cos a\sin h}{h}$$

$$= \lim_{h\to 0}\frac{-a^2\sin a\sin^2\left(\frac{h}{2}\right)}{\frac{h}{2}}\ +\ 2a\sin a\ +\ 0\ +\ a^2\cos a$$

$$\Rightarrow \lim_{h\to 0}\frac{(a\ +\ h)^2\sin(a\ +\ h)-a^2\sin a}{h} = 0\ +\ 2a\sin a\ +\ a^2\cos a$$

Hence, $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h} = 2a\sin a\ +\ a^2\cos a$

63. Question

Evaluate the following limits:

If $\lim_{x\to 0}kx\operatorname{cosec}x = \lim_{x\to 0}x\operatorname{cosec}kx$, find k.

Answer

Given, $\lim_{x\to 0}kx\operatorname{cosec}x\ =\ \lim_{x\to 0}x\operatorname{cosec}kx$

To Find: Value of k?

Explanation: Here, $\lim_{x\to 0}kx\operatorname{cosec}x\ =\ \lim_{x\to 0}x\operatorname{cosec}kx$

$$\lim_{x\to 0}kx\frac{1}{\sin x}\ =\ \lim_{x\to 0}x\frac{1}{\sin kx}$$

Taking k common from L.H.S and multiply and divide by k in R.H.S, we get

$$k\lim_{x\to 0}x\frac{1}{\sin x}\ =\ \frac{1}{k}\lim_{x\to 0}\frac{kx}{\sin kx}$$

$k = \frac{1}{k}$

$K^2 = 1$

$K = \pm 1$

Hence, The value of k is 1, - 1.

Exercise 29.8

1. Question

Evaluate the following limits:

$$\lim_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x$$

Answer

Given: $\lim_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x$

Assumption: Let $y = \frac{\pi}{2} - x$

So, $x \to \frac{\pi}{2}, y \to 0$

$$\Rightarrow \lim_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{y \to 0} y \tan\left(\frac{\pi}{2} - y\right)$$

$$\Rightarrow \lim_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{y \to 0} y \frac{\sin\left(\frac{\pi}{2} - y\right)}{\cos\left(\frac{\pi}{2} - y\right)}$$

$$\Rightarrow \lim_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{y \to 0} y \frac{\cos y}{\sin y}$$

$$\Rightarrow \lim_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{y \to 0} \cos y - \lim_{y \to 0} \frac{y}{\sin y}$$

$$\Rightarrow \lim_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x = \cos 0 - \frac{0}{\sin 0}$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x = 1 - 0$$

Hence, $\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) = 1$

2. Question

Evaluate the following limits:

$$\lim_{x \to \pi/2} \frac{\sin 2x}{\cos x}$$

Answer

Given, $\lim_{x \to \pi/2} \frac{\sin 2x}{\cos x}$

We know, sin2x = 2sin x.cos x

By putting this value, we get

$$\Rightarrow \lim_{x\to\pi/2} \frac{\sin 2x}{\cos x} = \lim_{x\to\pi/2} \frac{2\sin x\cos x}{\cos x}$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{\sin 2x}{\cos x} = \lim_{x\to\pi/2} 2\sin x$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{\sin 2x}{\cos x} = 2\sin\frac{\pi}{2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sin 2x}{\cos x} = 2\times 1$$

Hence $\lim_{x\to\pi/2} \frac{\sin 2x}{\cos x} = 2$

3. Question

Evaluate the following limits:

$$\lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x}$$

Answer

Given, $\lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x}$

Here, $\cos^2 x = 1-\sin^2 x$

By putting this we get,

$$\Rightarrow \lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x} = \lim_{x\to\pi/2} \frac{1-\sin^2 x}{1-\sin x}$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x} = \lim_{x\to\pi/2} \frac{(1-\sin x)(1+\sin x)}{1-\sin x}$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x} = \lim_{x\to\pi/2} 1+\sin x$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x} = 1+\sin\frac{\pi}{2}$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x} = 1+1$$

Hence, $\lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x} = 2$

4. Question

Evaluate the following limits:

$$\lim_{x\to\pi/3} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}(\pi/3-x)}$$

Answer

Given, $\lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

[Applying the formula 1 – cos 2x = $2\sin^2 x$]

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{2\sin^2 3x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{2}\sin 3x}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\sin 3x}{\left(\frac{\pi}{3}-x\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{3\sin 3x}{\pi-3x}$$

We know that, sin x = sin(π – x)

Therefore,

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x\to\frac{\pi}{3}} \frac{3\sin(\pi-3x)}{\pi-x}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = 3$$

Hence, $\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = 3$

5. Question

Evaluate the following limits:

$$\lim_{x\to a} \frac{\cos x-\cos a}{x-a}$$

Answer

Given, $\lim_{x\to a} \frac{\cos x-\cos a}{x-a}$

$$\Rightarrow \lim_{x\to a} \frac{\cos x-\cos a}{x-a} = \lim_{x\to a} \frac{\left(-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right)}{x-a}$$

$$\Rightarrow \lim_{x\to a} \frac{\cos x-\cos a}{x-a} = -2\lim_{x\to a}\sin\left(\frac{x+a}{a}\right)\lim_{x\to a}\sin\frac{\left(\frac{x-a}{a}\right)}{x-a}$$

$$\Rightarrow \lim_{x\to a} \frac{\cos x-\cos a}{x-a} = -2\sin\left(\frac{a+a}{a}\right)\left(\lim_{x\to a}\sin\frac{\left(\frac{x-a}{a}\right)}{x-a}\right)\times\frac{1}{2}$$

$$\Rightarrow \lim_{x\to a} \frac{\cos x-\cos a}{x-a} = -2\sin a \times 1\times\frac{1}{2}$$

$$\Rightarrow \lim_{x\to a} \frac{\cos x-\cos a}{x-a} = -\sin a$$

Hence, $\lim_{x\to a} \frac{\cos x-\cos a}{x-a} = -\sin a$

6. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}}$$

Answer

Given, $\lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}}$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}} = \lim_{y\to 0} \frac{1-\tan\left(y+\frac{\pi}{4}\right)}{y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}} = \lim_{y\to 0} \frac{\left(\frac{\tan y+\tan\frac{\pi}{4}}{1-\tan y\tan\frac{\pi}{4}}\right)}{y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}} = \lim_{y\to 0} \frac{(1-\tan y-\tan y-1)}{y(1-\tan y)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}} = \lim_{y\to 0} \frac{-2\tan y}{y(1-\tan y)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}} = -2\lim_{y\to 0} \frac{\tan y}{y} \times \lim_{y\to 0} \frac{1}{(1-\tan y)}$$

We know, $\lim_{x\to 0} \frac{\tan x}{x} = 1$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}} = -2 \times \frac{1}{(1-0)}$$

Hence, $\lim_{x\to\frac{\pi}{3}} \frac{1-\tan x}{x-\frac{\pi}{4}} = -2$

7. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$$

Answer

We have Given, If $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$

If $x \to \frac{\pi}{3}, \frac{\pi}{3} - x \to 0, \pi - 3x \to 0$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{1-\sin\left(\frac{\pi}{2}-y\right)}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{1-\cos y}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = 2\lim_{y\to 0}\left(\frac{\sin^2\frac{y}{2}}{\frac{y}{2}}\right)^2 \times \frac{1}{4}$$

Since, $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = 2\times 1\times\frac{1}{4}$$

Hence, $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = \frac{1}{2}$

8. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x}$$

Answer

We have $\lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x}$

If $x\to\frac{\pi}{3}, \frac{\pi}{3}-x\to 0, \pi-3x\to 0$

Let $\frac{\pi}{3}-x = y$ then $y\to 0$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x} = \lim_{y\to 0}\frac{\sqrt{3}-\frac{\tan\frac{\pi}{3}-\tan y}{1+\tan\frac{\pi}{3}.\tan y}}{3\left(\frac{\pi}{3}-x\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x} = \lim_{y\to 0}\frac{\sqrt{3}-\frac{\sqrt{3}-\tan y}{1+\sqrt{3}\tan y}}{3y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x} = \lim_{y\to 0}\frac{(\sqrt{3}+3\tan y-\sqrt{3}+\tan y}{3(1+\sqrt{3}\tan y)\,y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x} = \lim_{y\to 0}\frac{4\tan y}{3(1+\sqrt{3}\tan y)\,y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x} = \frac{4}{3}\lim_{y\to 0}\frac{\tan y}{y}\times\frac{1}{(1+\sqrt{3}\frac{\tan y}{y}\ y)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x} = \frac{4\times 1}{3}\times\frac{1}{1+0}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi-3x} = \frac{4}{3}$$

Hence, $\lim_{x\to\frac{\pi}{3}}\frac{\sqrt{3}-\tan x}{\pi-3x}=\frac{4}{3}$

9. Question

Evaluate the following limits:

$$\lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}$$

Answer

Given , $\lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}$

$$\Rightarrow \lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}=\lim_{x\to a}\frac{a\sin x-x\sin a}{ax(x-a)}$$

Let t = x - a

Then, as x→a, t→0

$$\Rightarrow \lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}=\lim_{t\to a}\frac{(a\sin(t+a)-(t+a)\sin a)}{a(t+a)t}$$

$$\Rightarrow \lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}=\lim_{t\to a}\frac{a\sin t\cos a+a\sin a(\cos t-1)-t\sin a}{a(t+a)t}$$

$$\Rightarrow \lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}=\lim_{t\to a}\frac{a\sin t\cos a+a\sin a\left(2\sin^2\left(\frac{t}{2}\right)\right)-t\sin a}{a(t+a)t}$$

$$\Rightarrow \lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}$$

$$=\lim_{t\to a}\frac{a\sin t\cos a}{a(t+a)t}+\lim_{t\to a}\frac{a\sin a\left(2\sin^2\left(\frac{t}{2}\right)\right)}{a(t+a)t}-\lim_{t\to a}\frac{t\sin a}{a(t+a)t}$$

$$\Rightarrow \lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}=\frac{a\cos a}{a^2}+-0-\frac{\sin a}{a^2}$$

$$\Rightarrow \lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}=\frac{a\cos a-\sin a}{a^2}$$

Hence, $\lim_{x\to a}\frac{a\sin x-x\sin a}{ax^2-xa^2}=\frac{a\cos a-\sin a}{a^2}$

10. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}$$

Answer

We have $\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}$

Rationalise the numerator, we get

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}\times\frac{\sqrt{2}+\sqrt{1+\sin x}}{\sqrt{2}+\sqrt{1+\sin x}}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{2-1-\sin x}{\cos^2 x\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{(1-\sin^2 x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{(1-\sin x)(1+\sin x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{1}{(1+\sin x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} = \frac{1}{(1+1)(\sqrt{2}+\sqrt{2})}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} = \frac{1}{4\sqrt{2}}$$

Hence, $\lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} = \frac{1}{4\sqrt{2}}$

11. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2}$$

Answer

Given, $\lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2}$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{\sqrt{2-\sin\left(\frac{\pi}{2}-y\right)}-1}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{\sqrt{2-\cos y}-1}{y^2}$$

Now, rationalize the Numerator, we get,

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{\sqrt{2-\cos y}-1}{y^2} \times \frac{\sqrt{2-\cos y}+1}{\sqrt{2-\cos y}+1}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{\sqrt{2-\cos y}-1}{y^2\left(\sqrt{2-\cos y}+1\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{2-\cos y-1}{y^2\left(\sqrt{2-\cos y}+1\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{1-\cos y}{y^2\left(\sqrt{2-\cos y}+1\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{y\to 0} \frac{2\sin^2\frac{y}{2}}{y^2\left(\sqrt{2-\cos y}+1\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = 2\times\lim_{y\to 0}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2\times\frac{1}{4}\times\frac{1}{\lim_{y\to 0}\sqrt{2-\cos y}+1}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = 2\times 1\times\frac{1}{4}\times\frac{1}{2}$$

Hence, $\lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2-\sin x}-1}{\left(\frac{\pi}{2}-x\right)^2} = \frac{1}{4}$

12. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2}$$

Answer

Given, $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2}$

Now, $x\to\frac{\pi}{4}, \frac{\pi}{4}-x\to 0$, let $\frac{\pi}{4}-x=y$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \lim_{y\to 0} \frac{\sqrt{2}-\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-y\right)}{y^2}$$

$$= \lim_{y\to 0} \frac{\sqrt{2}-\left[\cos\frac{\pi}{4}\cos y+\sin\frac{\pi}{4}\sin y+\sin\frac{\pi}{4}\cos y-\cos\frac{\pi}{4}\sin y\right]}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \lim_{y\to 0} \frac{\sqrt{2}-\left[\frac{\cos y}{\sqrt{2}}+\frac{\sin y}{\sqrt{2}}+\frac{\cos y}{\sqrt{2}}-\frac{\sin y}{\sqrt{2}}\right]}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \lim_{y\to 0} \frac{\sqrt{2}-\left[\frac{2\cos y}{\sqrt{2}}\right]}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \lim_{y\to 0} \frac{\sqrt{2}-\sqrt{2}\cos y}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \sqrt{2}\lim_{y\to 0} \frac{1-\cos y}{y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \sqrt{2}\lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{\frac{y^2}{4}}\times\frac{1}{4}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \sqrt{2}\times\frac{1}{4}\times\left(\lim_{y\to 0}\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \sqrt{2}\times 2\times\frac{1}{4}\times 1$$

Hence, $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)^2} = \frac{1}{\sqrt{2}}$

13. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3}$$

Answer

Given, $\lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3}$

Where $x\to\frac{\pi}{8}, \frac{\pi}{8}-x\to 0$, let $\frac{\pi}{8}-x=y$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3} = \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{8^3\left(\frac{\pi}{8}-x\right)^3}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3} = \lim_{y\to 0} \frac{\cot\left(\frac{\pi}{8}-x\right)4-\cos\left(\frac{\pi}{8}-x\right)4}{8^3y^3}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3} = \lim_{y\to 0} \frac{\tan 4y-\sin 4y}{8^3y^3}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3} = \lim_{y\to 0} \frac{\frac{\sin 4y}{\cos 4y}-\sin 4y}{8^3y^3}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3} = \lim_{y\to 0} \frac{\sin 4y-\sin 4y.\cos 4y}{8^3y^3\cos 4y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3} = \lim_{y\to 0} \frac{\sin 4y(1-\cos 4y)}{8^3y^3\cos 4y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3} = \lim_{y\to 0} \frac{\sin 4y.(2\sin^2 2y)}{8^3y^3\cos 4y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x-\cos 4x}{(\pi-8x)^3} = \frac{2}{8^3}\lim_{y\to 0} \frac{\sin 4y}{y}\times\frac{\sin^2 2y}{y^2}\times\frac{1}{\cos 4y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$$

$$= \frac{2}{8^3}\left(\lim_{y\to 0} \frac{\sin 4y}{4y} \times 4\right) \times \left(\lim_{y\to 0} \frac{\sin 2y}{2y} \times 2\right)^2 \times 4 \times \frac{1}{\lim\limits_{y\to 0} \cos 4y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \frac{2}{8^3}(1\times 4)\times(1)\times 4$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \frac{2\times 4\times 4}{8\times 8\times 8}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \frac{1}{16}$$

Hence, $\lim_{x\to\frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi-8x)^3} = \frac{1}{16}$

14. Question

Evaluate the following limits:

$$\lim_{x\to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}}$$

Answer

We have Given, $\lim_{x\to a} \frac{\cos x - \cos a}{\sqrt{x}-\sqrt{a}}$

$$\Rightarrow \lim_{x\to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = \lim_{x\to a} \frac{\left(-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right)}{\sqrt{x} - \sqrt{a}}$$

Now, Rationalize the Denominator

$$\Rightarrow \lim_{x\to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2\lim_{x\to a} \frac{\left(\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right)}{\sqrt{x} - \sqrt{a}(\sqrt{x}+\sqrt{a})} . \sqrt{x} + \sqrt{a}$$

$$\Rightarrow \lim_{x\to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2\lim_{x\to a} \sin\left(\frac{x+a}{2}\right) . \lim_{x\to a} \frac{\sin\frac{x-a}{2} \times \frac{1}{2}}{\frac{x-a}{2}} \lim_{x\to a} . \sqrt{x} + \sqrt{a}$$

$$\Rightarrow \lim_{x\to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2\sin(a) \times \frac{1}{2} \times 2\sqrt{a}$$

$$\Rightarrow \lim_{x\to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2\sqrt{a}\sin a$$

Hence, $\lim_{x\to a} \frac{\cos x - \cos a}{\sqrt{x}-\sqrt{a}} = -2\sqrt{a}\sin a$

15. Question

Evaluate the following limits:

$$\lim_{x\to\pi} \frac{\sqrt{5+\cos x} - 2}{(\pi - x)^2}$$

Answer

Given, $\lim_{x\to\pi} \frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}$

If x → π, then π – x → 0, let π – x = y

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=\lim_{y\to 0}\frac{\sqrt{5+\cos(\pi-y)}-2}{y^2}$$

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=\lim_{y\to 0}\frac{\sqrt{5+\cos y}-2}{y^2}$$

Rationalize the Numerator

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=\lim_{y\to 0}\frac{\sqrt{5+\cos y}-2\times(\sqrt{5+\cos y}+2)}{y^2(\sqrt{5+\cos y}+2)}$$

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=\lim_{y\to 0}\frac{5-\cos y-4}{y^2(\sqrt{5+\cos y}-2)}$$

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=\lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{y^2(\sqrt{5+\cos y}+2)}$$

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=2\times\lim_{y\to 0}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2\times\frac{1}{4}\times\frac{1}{\lim_{y\to 0}(\sqrt{5-\cos y}+2)}$$

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=2.\frac{1}{4},\frac{1}{\sqrt{4}+2}$$

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=2\times\frac{1}{4}\times\frac{1}{4}$$

$$\Rightarrow \lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=\frac{1}{8}$$

Hence, $\lim_{x\to\pi}\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}=\frac{1}{8}$

16. Question

Evaluate the following limits:

$$\lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}$$

Answer

We have Given, $\lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}$

$$\Rightarrow \lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}=\lim_{x\to a}\frac{\left(-2\sin\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)\right)}{\sqrt{x}-\sqrt{a}(\sqrt{x}+\sqrt{a})}$$

Now, Rationalize the Denominator

$$\Rightarrow \lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}=-2\lim_{x\to a}\frac{\left(\sin\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)\right)}{\frac{(\sqrt{x}-\sqrt{a})}{2}(\sqrt{x}+\sqrt{a})}\frac{1}{2}$$

$$\Rightarrow \lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a} = -2\sin(\sqrt{a})\times\frac{1}{2\sqrt{a}}\times\frac{1}{2}$$

$$\Rightarrow \lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a} = -\sin(\sqrt{a})\times\frac{1}{2\sqrt{a}}$$

Hence, $\lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a} = -\sin(\sqrt{a})\times\frac{1}{2\sqrt{a}}$

17. Question

Evaluate the following limits:

$$\lim_{x\to a}\frac{\sin\sqrt{x}-\sin\sqrt{a}}{x-a}$$

Answer

we have $\lim_{x\to a}\frac{\sin\sqrt{x}-\sin\sqrt{a}}{x-a}$

$$= \lim_{x\to a}\frac{\sin\sqrt{x}-\sin\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}$$

$$= \lim_{x\to a}\frac{2\sin\left(\frac{\sqrt{x}-a}{2}\right)\cos\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}$$

$$= 2\lim_{x\to a}\left[\frac{\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}{\frac{\sqrt{x}-\sqrt{a}}{2}}\right]\times\frac{1}{2}\times\lim_{x\to a}\left[\frac{\cos\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)}{\sqrt{x}+\sqrt{a}}\right]$$

$$= 2\times 1\times\frac{1}{2}\times\cos\sqrt{a}\times\frac{1}{2\sqrt{a}}$$

$$\Rightarrow \lim_{x\to a}\frac{\sin\sqrt{x}-\sin\sqrt{a}}{x-a} = \frac{\cos\sqrt{a}}{2\sqrt{a}}$$

18. Question

Evaluate the following limits:

$$\lim_{x\to 1}\frac{1-x^2}{\sin 2\pi x}$$

Answer

We have Given, $\lim_{x\to 1}\frac{1-x^2}{\sin 2\pi x}$

Here, x → 1, then x – 1 → 0, let x – 1 = y

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin 2\pi x} = \lim_{x-1\to 0}\frac{(1-x)(1+x)}{\sin 2\pi x}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin 2\pi x} = \lim_{x-1\to 0}\frac{(1-x)(1+x)}{\sin 2\pi x}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin 2\pi x} = \lim_{y\to 0}\frac{-y(1+y+1)}{\sin 2\pi(y+1)}$$

$$\Rightarrow \lim_{x \to 1} \frac{1-x^2}{\sin 2\pi x} = \lim_{y \to 0} \frac{-y(1+y+1)}{\sin 2\pi(y+1)}$$

$$\Rightarrow \lim_{x \to 1} \frac{1-x^2}{\sin 2\pi x} = \lim_{y \to 0} \frac{y(y+2)}{\sin 2\pi y + 2\pi}$$

$$\Rightarrow \lim_{x \to 1} \frac{1-x^2}{\sin 2\pi x} = \lim_{y \to 0} \frac{y(y+2)}{\sin 2\pi y}$$

$$\Rightarrow \lim_{x \to 1} \frac{1-x^2}{\sin 2\pi x} = \lim_{y \to 0} (y+2) \times \frac{y}{\sin \frac{2\pi y}{2\pi y} \times 2\pi y}$$

$$\Rightarrow \lim_{x \to 1} \frac{1-x^2}{\sin 2\pi x} = -2 \times \frac{1}{2\pi}$$

$$\Rightarrow \lim_{x \to 1} \frac{1-x^2}{\sin 2\pi x} = -\frac{1}{\pi}$$

Hence, $\lim_{x \to 1} \frac{1-x^2}{\sin 2\pi x} = -\frac{1}{\pi}$

19. Question

Evaluate the following limits:

$$\lim_{x \to \frac{\pi}{4}} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$$, where f(x) = sin 2x

Answer

Given, f(x) = sin 2x

Since, $\lim_{x \to \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$

$$\Rightarrow \lim_{x \to \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{x \to \frac{\pi}{4}} \frac{\sin 2x - \sin\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{4}}$$

$$\Rightarrow \lim_{x \to \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{x \to \frac{\pi}{4}} \frac{\sin 2x - \sin\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{4}}$$

Now, $x \to \frac{\pi}{4}$ and $x - \frac{\pi}{4} \to 0$, let $x - \frac{\pi}{4} = y$

$$\Rightarrow \lim_{x \to \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{y \to 0} \frac{\sin 2\left(y + \frac{\pi}{4}\right) - 1}{y}$$

$$\Rightarrow \lim_{x \to \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{y \to 0} \frac{\sin\left(\frac{\pi}{2} + 2y\right) - 1}{y}$$

$$\Rightarrow \lim_{x \to \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{y \to 0} \frac{\cos 2y - 1}{y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{y\to 0} \frac{1 - \cos 2y}{y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = -\lim_{y\to 0} \frac{2\sin^2 y}{y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = -2\lim_{y\to 0} \left(\frac{\sin y}{y}\right)^2 \times y$$

Since, $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = -2 \times 0$$

Hence, $\lim_{x\to\frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = 0$

20. Question

Evaluate the following limits:

$$\lim_{x\to 1} \frac{1 + \cos \pi x}{(1 - x)^2}$$

Answer

Given, $\lim_{x\to 1} \frac{1+\cos \pi x}{(1-x)^2}$

Now, $x \to 1$, then $x - 1 \to 0$, let $x - 1 = y$

$$\Rightarrow \lim_{x\to 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \lim_{y\to 0} \frac{1 + \cos \pi(y + 1)}{-y^2}$$

$$\Rightarrow \lim_{x\to 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \lim_{y\to 0} \frac{1 + \cos \pi(y + 1)}{y^2}$$

$$\Rightarrow \lim_{x\to 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \lim_{y\to 0} \frac{1 - \cos(\pi y)}{y^2}$$

$$\Rightarrow \lim_{x\to 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \lim_{y\to 0} \frac{2\sin^2 \frac{\pi y}{2}}{y^2}$$

$$\Rightarrow \lim_{x\to 1} \frac{1 + \cos \pi x}{(1 - x)^2} = 2\lim_{y\to 0} \left(\frac{\sin \frac{\pi y}{2}}{\frac{\pi y}{2}}\right)^2 \times \frac{\pi^2}{4}$$

$$\Rightarrow \lim_{x\to 1} \frac{1 + \cos \pi x}{(1 - x)^2} = 2 \times 1 \times \frac{\pi^2}{4}$$

Hence, $\lim_{x\to 1} \frac{1+\cos \pi x}{(1-x)^2} = \frac{\pi^2}{2}$

21. Question

Evaluate the following limits:

$$\lim_{x\to 1}\frac{1-x^2}{\sin \pi x}$$

Answer

We have Given, $\lim_{x\to 1}\frac{1-x^2}{\sin \pi x}$

Here, x → 1, then x – 1 → 0, let x – 1 = y

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin \pi x} = \lim_{x-1\to 0}\frac{(1-x)(1+x)}{\sin \pi x}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin \pi x} = \lim_{x-1\to 0}\frac{(1-x)(1+x)}{\sin \pi x}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin \pi x} = \lim_{y\to 0}\frac{-y(1+y+1)}{\sin \pi(y+1)}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin \pi x} = \lim_{y\to 0}\frac{y(y+2)}{\sin \pi y + \pi}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin \pi x} = \lim_{y\to 0}\frac{y(y+2)}{\frac{\sin \pi y}{2}}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin \pi x} = \lim_{y\to 0}\frac{y+2}{\frac{\sin \pi y}{\pi y}\pi y}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-x^2}{\sin \pi x} = \frac{2}{\pi}$$

Hence, $\lim_{x\to 1}\frac{1-x^2}{\sin 2\pi x} = \frac{2}{\pi}$

22. Question

Evaluate the following limits:

$$\lim_{x\to \frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x}$$

Answer

We have $\lim_{x\to \frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x}$

$$\Rightarrow \lim_{x\to \frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x} = \lim_{y\to 0}\frac{\left(1-\sin 2\left(y+\frac{\pi}{4}\right)\right)}{1+\cos 4\left(y+\frac{\pi}{4}\right)}$$

$$\Rightarrow \lim_{x\to \frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x} = \lim_{y\to 0}\frac{\left(1-\sin\left(\frac{\pi}{2}+2y\right)\right)}{1+\cos(\pi+4y)}$$

$$\Rightarrow \lim_{x\to \frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x} = \lim_{y\to 0}\frac{1-\cos 2y}{1-\cos 4y}$$

$$\Rightarrow \lim_{x\to \frac{\pi}{4}}\frac{1-\sin 2x}{1+\cos 4x} = \lim_{y\to 0}\frac{2\sin^2 y}{2\sin^2 2y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\sin 2x}{1+\cos 4x} = \lim_{y\to 0} \frac{\left(\frac{2\sin^2 y}{y}\right)^2 y^2}{\left(\frac{2\sin^2 2y}{2y}\right)^2 4y^2}$$

Since, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, then

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\sin 2x}{1+\cos 4x} = \frac{1\times y^2}{1\times 4y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\sin 2x}{1+\cos 4x} = \frac{1}{4}$$

Hence, $\lim_{x\to\frac{\pi}{4}} \frac{1-\sin 2x}{1+\cos 4x} = \frac{1}{4}$

23. Question

Evaluate the following limits:

$$\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$$

Answer

Given, $\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$

As we know, $\tan^2 x = \sec^2 x - 1$

$$\Rightarrow \lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi} \frac{1+\cos x}{\sec^2 x - 1}$$

$$\Rightarrow \lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi} \frac{1+\cos x}{\frac{1}{\cos^2 x} - 1}$$

$$\Rightarrow \lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi} \frac{\cos^2 x.(1+\cos x)}{1-\cos^2 x}$$

$$\Rightarrow \lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi} \frac{\cos^2 x.(1+\cos x)}{(1+\cos x)(1-\cos x)}$$

$$\Rightarrow \lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi} \frac{\cos^2 x}{1-\cos x}$$

$$\Rightarrow \lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \frac{1}{1-(-1)}$$

Hence, $\Rightarrow \lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \frac{1}{2}$

24. Question

Evaluate the following limits:

$$\lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)$$

Answer

$$\lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)$$

Divide and multiply by 2, we get

$$\Rightarrow \lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right) = \lim_{n\to\infty} 2\left[n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)\right]\times\frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right) = \lim_{n\to\infty} n\sin\frac{\pi}{2n}\times\frac{1}{2}$$

Now, $n\to\infty$, then $\frac{1}{n}\to 0$, let $\frac{1}{n} = y$

$$\Rightarrow \lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right) = \lim_{y\to 0}\frac{1}{y}\sin\frac{\pi}{2}\times\frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right) = \frac{1}{2}\lim_{y\to 0}\frac{\left(\sin\left(\frac{\pi}{2}\right)y\right)}{y}$$

$$\Rightarrow \lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right) = \frac{1}{2}\lim_{y\to 0}\frac{\left(\sin\left(\frac{\pi}{2}\right)y\right)}{\frac{\pi y}{2}}\times\frac{\pi}{2}$$

$$\Rightarrow \lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right) = \frac{1}{2}\times\frac{1\pi}{2}$$

$$\Rightarrow \lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right) = \frac{\pi}{4}$$

Hence, $\lim_{n\to\infty} n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right) = \frac{\pi}{4}$

25. Question

Evaluate the following limits:

$$\lim_{n\to\infty} 2^{n-1}\sin\left(\frac{a}{2^n}\right)$$

Answer

We have $\lim_{n\to\infty} 2^{n-1}\sin\left(\frac{a}{2^n}\right)$

$$\Rightarrow \lim_{n\to\infty} 2^{n-1}\sin\left(\frac{a}{2^n}\right) = \lim_{n\to\infty}\frac{2^n}{2^1}\sin\left(\frac{a}{2^n}\right)$$

$$\Rightarrow \lim_{n\to\infty} 2^{n-1}\sin\left(\frac{a}{2^n}\right) = \lim_{n\to\infty}\frac{2^n}{2^1}\sin\frac{a}{2^n}$$

Now, $n\to\infty$, $\frac{1}{n}\to 0$ and let $h = 1/n$

$$\Rightarrow \lim_{n\to\infty} 2^{n-1}\sin\left(\frac{a}{2^n}\right) = \lim_{h\to 0}\frac{2^{\frac{1}{h}}}{2^1}.\sin\frac{a}{2^{\frac{1}{h}}}$$

$$\Rightarrow \lim_{n\to\infty} 2^{n-1}\sin\left(\frac{a}{2^n}\right) = \lim_{h\to 0}\frac{2^{\frac{1}{h}}}{2^1}.\frac{\left(\sin\frac{a}{2^{\frac{1}{h}}}\right)}{\frac{a}{2^{\frac{1}{h}}}}.\frac{a}{2^{\frac{1}{h}}}$$

We know, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ then, we get

$$\Rightarrow \lim_{n\to\infty} 2^{n-1}\sin\left(\frac{a}{2^n}\right) = \frac{a}{2}$$

Hence, $\lim_{n\to\infty} 2^{n-1}\sin\left(\frac{a}{2^n}\right) = \frac{a}{2}$

26. Question

Evaluate the following limits:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)}$$

Answer

We have $\lim_{n\to\infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)}$

Now, $n \to \infty, \frac{1}{n} = h \to 0$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} = \frac{\lim_{h\to 0}\sin\left(\frac{a}{2^{\frac{1}{h}}}\right)}{\lim_{h\to 0}\sin\left(\frac{b}{2^{\frac{1}{h}}}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} = \frac{\lim_{h\to 0}\sin\frac{\left(\frac{a}{2^{\frac{1}{h}}}\right)}{\frac{a}{2^{\frac{1}{h}}}}\cdot\frac{a}{2^{\frac{1}{h}}}}{\lim_{h\to 0}\sin\frac{\left(\frac{b}{2^{\frac{1}{h}}}\right)}{\frac{b}{2^{\frac{1}{h}}}}\cdot\frac{b}{2^{\frac{1}{h}}}}$$

We know, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ then , we get

$$\Rightarrow \lim_{n\to\infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} = \frac{1\times\frac{a}{2^{\frac{1}{h}}}}{1\times\frac{b}{2^{\frac{1}{h}}}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} = \frac{a}{b}$$

Hence, $\lim_{n\to\infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} = \frac{a}{b}$

27. Question

Evaluate the following limits:

$$\lim_{x\to -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x+1)}$$

Answer

We have $\lim_{x\to -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x+1)}$

$$\Rightarrow \lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \lim_{x\to-1}\frac{x^2-x-2}{x(x+1)+\sin(x+1)}$$

$$\Rightarrow \lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \lim_{x\to-1}\frac{(x-2)(x+1)}{x(x+1)+\sin(x+1)}$$

$$\Rightarrow \lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \lim_{x\to-1}\frac{1}{\frac{x(x+1)}{(x-2)(x+1)}+\frac{\sin(x+1)}{(x-2)(x+1)}}$$

$$\Rightarrow \lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \lim_{x\to-1}\frac{1}{\frac{x}{(x-2)}+\frac{\sin(x+1)}{(x-2)(x+1)}}$$

$$\Rightarrow \lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \lim_{x\to-1}\frac{1}{(x-2)}\left[\frac{1}{x+\frac{\sin(x+1)}{(x+1)}}\right]$$

$$\Rightarrow \lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \lim_{x\to-1}\frac{1}{(x-2)}\left[\frac{1}{\lim_{x\to-1}x+\lim_{x\to-1}\frac{\sin(x+1)}{(x+1)}}\right]$$

$$\Rightarrow \lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \left(\frac{1}{-1-2}\right)\times\left(\frac{1}{(-1)+1}\right)$$

$$\Rightarrow \lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \frac{1}{0} = \infty$$

Hence, $\lim_{x\to-1}\frac{x^2-x-2}{(x^2+x)+\sin(x+1)} = \infty$

28. Question

Evaluate the following limits:

$$\lim_{x\to2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)}$$

Answer

We have $\lim_{x\to2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)}$

$$\Rightarrow \lim_{x\to2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)} = \lim_{x\to2}\frac{(x-2)(x+1)}{x^2-2x+\sin(x-2)}$$

$$\Rightarrow \lim_{x\to2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)} = \lim_{x\to2}\frac{1}{\frac{x}{x+1}+\frac{\sin(x-2)}{(x-2)(x+1)}}$$

$$\Rightarrow \lim_{x\to2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)} = \lim_{x\to2}(x+1)\left[\frac{1}{x+\frac{\sin(x-2)}{(x-2)}}\right]$$

$$\Rightarrow \lim_{x\to2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)} = \lim_{x\to2}(x+1)\left[\frac{1}{\lim_{x\to2}(x)+\lim_{x\to2}\frac{\sin(x-2)}{(x-2)}}\right]$$

$$\Rightarrow \lim_{x\to 2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)} = (2+1)\left[\frac{1}{2+\lim\limits_{x\to 2}\frac{\sin(x-2)}{(x-2)}}\right]$$

$$\Rightarrow \lim_{x\to 2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)} = 3\left[\frac{1}{2+1}\right]$$

Hence, $\lim\limits_{x\to 2}\frac{x^2-x-2}{x^2-2x+\sin(x-2)} = 1$

29. Question

Evaluate the following limits:

$$\lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)$$

Answer

We have $\lim\limits_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)$

When, $x \to 1, x-1 \to 0$, let $x-1-y$, then $y \to 0$

$$\Rightarrow \lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) = \lim_{(x-1)\to 0} -(x-1)\tan\left(\frac{\pi x}{2}\right)$$

$$\Rightarrow \lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) = -\lim_{y\to 0} y\tan\left(\frac{\pi}{2}\right)(y+1)$$

$$\Rightarrow \lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) = -\lim_{y\to 0} y\tan\left(\frac{\pi}{2}+\frac{\pi}{2}y\right)$$

$$\Rightarrow \lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) = \lim_{y\to 0} y\left(\cot\frac{\pi}{2}y\right)$$

$$\Rightarrow \lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) = \lim_{y\to 0}\frac{y}{\tan\frac{\pi y}{2}}$$

$$\Rightarrow \lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) = \lim_{y\to 0}\frac{\frac{\pi y}{2}\times\frac{2}{\pi}}{\tan\frac{\pi y}{2}}$$

$$\Rightarrow \lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) = \frac{2}{\pi}$$

Hence, $\lim\limits_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) = \frac{2}{\pi}$

30. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}$$

Answer

We have $\lim\limits_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}$

If $x\to\frac{\pi}{4}$, then $x-\frac{\pi}{4}=0$, let $x-\frac{\pi}{4}\to y$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{1-\tan y}{1-\sqrt{2}\sin y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{1-\tan\left(y+\frac{\pi}{4}\right)}{1-\sqrt{2}\sin\left(y+\frac{\pi}{4}\right)}$$

Since, $\tan(a+b) = \frac{\tan a+\tan b}{1=\tan a.\tan b}$

$\sin(a+b) = \sin a.\cos b + \cos a.\sin b$

By putting these , we get

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{1-\left(\frac{\tan\frac{\pi}{4}+\tan y}{1+\tan\frac{\pi}{4}.\tan y}\right)}{1-\sqrt{2}\left(\cos\frac{\pi}{4}+\cos y.\sin\frac{\pi}{4}\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{\left(1-\left(\frac{1+\tan y}{1-\tan y}\right)\right)}{1-\sqrt{2}\left(\frac{\sin y}{\sqrt{2}}+\frac{\cos y}{\sqrt{2}}\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{1-\tan y-1-\tan y}{(1-\tan y)(1-\sin y+\cos y)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{-2\tan y}{(1-\tan y)(1-\sin y+\cos y)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = -2\lim_{y\to 0} \frac{\tan y\times 1}{(1-\tan y)(1-\sin y+\cos y)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{\tan y\times 1}{\lim_{y\to 0}(1-\tan y)\ \lim_{y\to 0}(1-\sin y-\cos y)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{\lim_{y\to 0}\left(\frac{\tan y}{y}\right)\times y}{\lim_{y\to 0}(1-\tan y)\ \lim_{y\to 0}(1-\sin y-\cos y)}$$

Since, $\frac{\tan y}{y} = 1$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = -\lim_{y\to 0} \frac{2y}{(1-y)(1-y-1)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{y\to 0} \frac{2}{1-y}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = 2$$

Hence, $\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = 2$

31. Question

Evaluate the following limits:

$$\lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi - x)^2}$$

Answer

We have Given, $\lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$

If $x \to \pi$, then $x - \pi = 0$, let $x - \pi \to y$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{x-\pi \to 0} \frac{\sqrt{2+\cos x}-1}{(-1)^2(x-\pi)^2}$$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{y \to 0} \frac{\sqrt{2+\cos(\pi+y)}-1}{y^2}$$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{y \to 0} \frac{\sqrt{2-\cos y}-1}{y^2}$$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{y \to 0} \frac{(\sqrt{2-\cos y}-1)(\sqrt{2-\cos y}-1)}{y^2(\sqrt{2-\cos y}+1)}$$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{y \to 0} \frac{1-\cos y}{y^2(\sqrt{2-\cos y}+1)}$$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{y \to 0} \frac{2\sin^2\frac{y}{2}}{y^2(\sqrt{2-\cos y}+1)}$$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = 2.\lim_{y \to 0} \left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 \times \frac{1}{4} \times \frac{1}{\lim_{y \to 0} \sqrt{2-\cos 0}+1)}$$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = 2 \times \frac{1}{4} \times \frac{1}{\sqrt{2-1}+1}$$

$$\Rightarrow \lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \frac{1}{4}$$

Hence, $\lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \frac{1}{4}$

32. Question

Evaluate the following limits:

$$\lim_{x \to \pi/4} \frac{\sqrt{\cos x}-\sqrt{\sin x}}{x-\pi/4}$$

Answer

$$\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{\cos x}-\sqrt{\sin x}}{x-\frac{\pi}{4}}$$

Rationalizing we get,

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{\cos x}-\sqrt{\sin x}}{x-\frac{\pi}{4}} \times \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\left(x - \frac{\pi}{4}\right)\left(\sqrt{\cos x} + \sqrt{\sin x}\right)}$$

As, $x \to \frac{\pi}{4}, x - \frac{\pi}{4} \to 0, \text{let } x - \frac{\pi}{4} = y$

Therefore, y → 0,

Now,

$$= \lim_{y\to 0} \frac{\left(\cos\left(\frac{\pi}{4} + y\right) - \sin\left(\frac{\pi}{4} + y\right)\right)}{y\left(\sqrt{\cos\left(\frac{\pi}{4} + y\right)} + \sqrt{\sin\left(\frac{\pi}{4} + y\right)}\right)}$$

$$= \lim_{y\to 0} \frac{\frac{1}{\sqrt{2}}\cos y - \frac{1}{\sqrt{2}}\sin y - \left[\frac{1}{\sqrt{2}}\cos y + \frac{1}{\sqrt{2}}\sin y\right]}{y\left(\sqrt{\frac{1}{\sqrt{2}}\cos y - \frac{1}{\sqrt{2}}\sin y} + \sqrt{\frac{1}{\sqrt{2}}\cos y + \frac{1}{\sqrt{2}}\sin y}\right)}$$

$$= \lim_{y\to 0} \frac{-\sqrt{2}\sin y}{y\left[\sqrt{\frac{1}{\sqrt{2}}\cos y - \frac{1}{\sqrt{2}}\sin y} + \sqrt{\frac{1}{\sqrt{2}}\cos y + \frac{1}{\sqrt{2}}\sin y}\right]}$$

$$= \frac{-1}{\frac{1}{2^{\frac{1}{4}}}}$$

Hence, $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{\cos x} - \sqrt{\sin x}}{x - \frac{\pi}{4}} = -2^{\frac{1}{4}}$

33. Question

Evaluate the following limits:

$$\lim_{x\to 1} \frac{1 - \frac{1}{x}}{\sin \pi(x - 1)}$$

Answer

We have Given, $\lim_{x\to 1} \frac{1-\frac{1}{x}}{\sin \pi(x-1)}$

if x → 1 then, x – 1 → 0 let x – 1 = y

$$\Rightarrow \lim_{x\to 1} \frac{1 - \frac{1}{x}}{\sin \pi (x - 1)} = \lim_{x-1\to 0} \frac{x - 1}{x \sin \pi (x - 1)}$$

$$\Rightarrow \lim_{x\to 1} \frac{1 - \frac{1}{x}}{\sin \pi (x - 1)} = \lim_{y\to 0} \frac{y}{\sin \pi y(y + 1)}$$

$$\Rightarrow \lim_{x\to 1} \frac{1 - \frac{1}{x}}{\sin \pi (x - 1)} = \lim_{y\to 0} \frac{1}{\frac{\sin \pi y(y + 1)}{y}}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-\frac{1}{x}}{\sin\pi(x-1)} = \frac{1}{\lim_{y\to 0}(y+1)\times\lim_{y\to 0}\left(\frac{\sin\pi y}{y\times\pi}.\pi\right)}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-\frac{1}{x}}{\sin\pi(x-1)} = \frac{1}{(1)(1\times\pi)}$$

$$\Rightarrow \lim_{x\to 1}\frac{1-\frac{1}{x}}{\sin\pi(x-1)} = \frac{1}{\pi}$$

Hence, $\lim_{x\to 1}\frac{1-\frac{1}{x}}{\sin\pi(x-1)} = \frac{1}{\pi}$

34. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{6}}\frac{\cot^2 x-3}{\operatorname{cosec} x-2}$$

Answer

[$\operatorname{cosec}^2 x$ – $\cot^2 x$ = 1]

$$\lim_{x\to\frac{\pi}{6}}\frac{\cot^2 x-3}{\operatorname{cosec}x-2} = \lim_{x\to\frac{\pi}{6}}\frac{\operatorname{cosec}^2 x-4}{\operatorname{cosec}x-1}$$

[Applying, $a^2 - b^2 = (a + b)(a - b)$]

$$\lim_{x\to\frac{\pi}{6}}\frac{\cot^2 x-3}{\operatorname{cosec}x-2} = \lim_{x\to\frac{\pi}{6}}\frac{(\operatorname{cosec}x+2)(\operatorname{cosec}x-2)}{\operatorname{cosec}x-2}$$

Hence, $\lim_{x\to\frac{\pi}{6}}\frac{\cot^2 x-3}{\operatorname{cosec}x-2} = 2+2=4$

35. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}$$

Answer

We have Given , $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}$

Now, if $x\to\frac{\pi}{4}$ then $x-\frac{\pi}{4}\to 0$ let $x-\frac{\pi}{4}\to y$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \lim_{x-\frac{\pi}{4}\to 0}\frac{\sqrt{2}-\cos x-\sin x}{(4)^2\left(x-\frac{\pi}{4}\right)^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \lim_{y\to 0}\frac{\sqrt{2}-\cos\left(y+\frac{\pi}{4}\right)-\sin\left(y+\frac{\pi}{4}\right)}{16y^2}$$

Here, cos(a+b) =cos a.cos b – sin a.sin b

And, sin(a+b) = sin a.cos b+cos a.sin b

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \lim_{y\to 0}\frac{\sqrt{2}-\left(\cos y\cos\frac{\pi}{4}-\sin y\sin\frac{\pi}{4}\right)-\left(\sin y\cos\frac{\pi}{4}+\cos y\sin\frac{\pi}{4}\right)}{16y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \lim_{y\to 0}\frac{\sqrt{2}-\left(\cos y.\frac{1}{\sqrt{2}}-\sin y.\frac{1}{\sqrt{2}}\right)-\left(\sin y.\frac{1}{\sqrt{2}}+\cos y.\frac{1}{\sqrt{2}}\right)}{16y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \lim_{y\to 0}\frac{\sqrt{2}-\frac{1}{\sqrt{2}}(\cos y-\sin y)-\frac{1}{\sqrt{2}}(\sin y+\cos y)}{16y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \lim_{y\to 0}\frac{\sqrt{2}-\frac{1}{\sqrt{2}}[(\cos y-\sin y)-(\sin y+\cos y)]}{16y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \lim_{y\to 0}\frac{\sqrt{2}-\frac{1}{\sqrt{2}}[(\cos y-\sin y)-(\sin y+\cos y)]}{16y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \lim_{y\to 0}\frac{\sqrt{2}(1-\cos y)}{16y^2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \frac{\sqrt{2}}{8}\lim_{y\to 0}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2\times\frac{1}{4}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \frac{1}{16\sqrt{2}}$$

Hence, $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2} = \frac{1}{16\sqrt{2}}$

36. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x-2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}$$

Answer

We have Given, $\lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x-2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x-2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}=\lim_{y\to 0}\frac{\left(y\sin\left(\frac{\pi}{2}-y\right)-2\cos\left(\frac{\pi}{2}-y\right)\right)}{y+\cot\left(\frac{\pi}{2}-y\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x-2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}=\lim_{y\to 0}\left(\frac{y\cos y-2\sin y}{1+\tan y}\right)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x-2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}=\lim_{y\to 0}\left(\frac{\cos y-2.\frac{\sin y}{y}}{1+\frac{\tan y}{y}}\right)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x-2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}=\frac{1-2}{1+1}=-\frac{1}{2}$$

Hence, $\lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x-2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}=-\frac{1}{2}$

37. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}$$

Answer

We have Given, $\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}$

if $x\to\frac{\pi}{4}$ then $x-\frac{\pi}{4}\to 0$ let $x-\frac{\pi}{4}=y$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}=\lim_{y\to 0}\frac{\cos\left(\frac{\pi}{4}+y\right)-\sin\left(\frac{\pi}{4}+y\right)}{-y\left(\cos\left(\frac{\pi}{4}+y\right)+\sin\left(\frac{\pi}{4}+y\right)\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}$$
$$=\lim_{y\to 0}\frac{\left(\cos\frac{\pi}{4}\cos y-\sin\frac{\pi}{4}\sin y\right)-\left(\sin\frac{\pi}{4}\cos y+\cos\frac{\pi}{4}\sin y\right)]}{-y\left(\cos\left(\frac{\pi}{4}+y\right)+\sin\left(\frac{\pi}{4}+y\right)\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}=\lim_{y\to 0}\frac{\frac{\cos}{\sqrt{2}}-\frac{\sin}{\sqrt{2}}-\frac{\cos}{\sqrt{2}}-\frac{\sin}{\sqrt{2}}}{-y\left(\cos\left(\frac{\pi}{4}+y\right)+\sin\left(\frac{\pi}{4}+y\right)\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}=\lim_{y\to 0}\frac{-\frac{2\sin y}{\sqrt{2}}}{-y\left(\cos\left(\frac{\pi}{4}+y\right)+\sin\left(\frac{\pi}{4}+y\right)\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)}$$

$$= \sqrt{2} \lim_{y\to 0} \left(\frac{\sin y}{y}\right) \frac{1}{\lim_{y\to 0}\left(\cos\left(\frac{\pi}{4} + y\right) + \sin\left(\frac{\pi}{4} + y\right)\right)}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)} = \sqrt{2} \times 1 \times \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)} = \sqrt{2} \times \frac{1}{\frac{2}{\sqrt{2}}}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)} = \frac{\sqrt{2} \times \sqrt{2}}{2}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)} = 1$$

Hence, the answer is 1.

38. Question

Evaluate the following limits:

$$\lim_{x\to\pi} \frac{1 - \sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)}$$

Answer

$$= \lim_{x\to\pi} \frac{1 - \sin\left(\frac{x}{2}\right)}{\left(\cos^2\left(\frac{x}{4}\right) - \sin^2\frac{x}{4}\right)\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)}$$

$$= \lim_{x\to\pi} \frac{1 - \sin\left(\frac{x}{2}\right)}{\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)^2\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)}$$

$$= \lim_{x\to\pi} \frac{1 - \sin\left(\frac{x}{2}\right)}{\left(1 - \sin\frac{x}{2}\right)\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)}$$

$$= \lim_{x\to\pi} \frac{1}{\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)}$$

$$= \frac{\sqrt{2}}{2}$$

$$= \frac{1}{\sqrt{2}}$$

Hence,

$$\lim_{x\to\pi} \frac{1 - \sin\left(\frac{x}{2}\right)}{\left(\cos\frac{x}{2}\right)\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)} = \frac{1}{\sqrt{2}}$$

Exercise 29.9

1. Question

Evaluate the following limits:

$$\lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x}$$

Answer

As we need to find $\lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi}\frac{1+\cos\pi}{\tan^2\pi} = \frac{1-1}{0} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

Tip: Similar limit problems involving trigonometric ratios are mostly solved using sandwich theorem.
$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$

So to solve this problem we need to have a sin term so that we can make use of sandwich theorem.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

As, $Z = \lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x}$

Multiplying numerator and denominator by 1-cos x, We have-

$$Z = \lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x}\times\frac{1-\cos x}{1-\cos x}$$

$\Rightarrow Z = \lim_{x\to\pi}\frac{1-\cos^2 x}{\tan^2 x(1-\cos x)}$

{As $1-\cos^2x = \sin^2x$}

$\Rightarrow Z = \lim_{x\to\pi}\frac{\sin^2 x}{\tan^2 x(1-\cos x)}$

$\Rightarrow Z = \lim_{x\to\pi}\frac{1}{1-\cos x}\times\lim_{x\to\pi}\frac{\sin^2 x}{\tan^2 x}$

$$\Rightarrow Z = \lim_{x\to\pi}\frac{1}{1-\cos\pi}\times\lim_{x\to\pi}\frac{\sin^2 x}{\tan^2 x}$$

$$\Rightarrow Z = \frac{1}{2}\lim_{x\to\pi}\frac{\sin^2 x}{\tan^2 x}$$

To apply sandwich theorem, we need to have limit such that variable tends to 0 and following forms should be there $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$

Here x→ π so we need to do modifications before applying the theorem.

As, sin (π-x) = sin x or sin (x - π) = -sin x and tan(π – x) = -tan x

∴ we can say that-

$\sin^2x = \sin^2(x-\pi)$ and $\tan^2x = \tan^2(x-\pi)$

As x → π

∴ (x – π) → 0

Let us represent x - π with y

$$\therefore Z = \frac{1}{2}\lim_{(x-\pi)\to 0}\frac{\sin^2(x-\pi)}{\tan^2(x-\pi)} = \frac{1}{2}\lim_{y\to 0}\frac{\sin^2 y}{\tan^2 y}$$

Dividing both numerator and denominator by y^2

$$Z = \frac{1}{2}\lim_{y\to 0}\frac{\frac{(\sin^2 y)}{y^2}}{\frac{\tan^2 y}{y^2}}$$

$$\Rightarrow Z = \frac{1}{2}\frac{\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2}{\lim_{y\to 0}\left(\frac{\tan y}{y}\right)^2}$$ {Using basic limits algebra}

As, $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$

$$\therefore Z = \frac{1}{2}\times\frac{1}{1} = \frac{1}{2}$$

$$\therefore \lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x} = \frac{1}{2}$$

2. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{4}}\frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$$

Answer

As we need to find $\lim_{x\to\frac{\pi}{4}}\frac{\operatorname{cosec}^2 x-2}{\cot x-1}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to\frac{\pi}{4}}\frac{\operatorname{cosec}^2 x-2}{\cot x-1} = \lim_{x\to\frac{\pi}{4}}\frac{\operatorname{cosec}^2\left(\frac{\pi}{4}\right)-2}{\cot\frac{\pi}{4}-1} = \frac{(\sqrt{2})^2-2}{1-1} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

As $Z = \lim_{x\to\frac{\pi}{4}}\frac{\operatorname{cosec}^2 x-2}{\cot x-1}$

$$\Rightarrow Z = \lim_{x\to\frac{\pi}{4}}\frac{\operatorname{cosec}^2 x\left(1-\frac{2}{\operatorname{cosec}^2 x}\right)}{\cot x\left(1-\frac{1}{\cot x}\right)} = \lim_{x\to\frac{\pi}{4}}\frac{\operatorname{cosec}^2 x(1-2\sin^2 x)}{\cot x\,(1-\tan x)}$$

$$\because \cot x = \frac{\operatorname{cosec} x}{\sec x}$$

$$\therefore Z = \lim_{x\to\frac{\pi}{4}}\frac{\sec x\ \operatorname{cosec} x(1-2\sin^2 x)}{1-\tan x}$$

$$\Rightarrow Z = \lim_{x\to\frac{\pi}{4}}(\sec x\operatorname{cosec} x)\times\lim_{x\to\frac{\pi}{4}}\left(\frac{1-2\sin^2 x}{1-\tan x}\right)$$

{Using basic limits algebra}

$$\Rightarrow Z = \sec\frac{\pi}{4}\ \operatorname{cosec}\frac{\pi}{4}\times\lim_{x\to\frac{\pi}{4}}\left(\frac{1-2\sin^2 x}{1-\tan x}\right) = 2\times\lim_{x\to\frac{\pi}{4}}\left(\frac{1-2\sin^2 x}{1-\tan x}\right)$$

$\because$ (1- 2sin^{2}x) = cos 2x $= \frac{1-\tan^2 x}{1+\tan^2 x}$

$$\therefore\ Z = 2 \times \lim_{x \to \frac{\pi}{4}} \left(\frac{\frac{1-\tan^2 x}{1+\tan^2 x}}{1-\tan x} \right)$$

$$\Rightarrow\ Z = 2 \times \lim_{x \to \frac{\pi}{4}} \left(\frac{1-\tan^2 x}{(1-\tan x)(1+\tan^2 x)} \right)$$

As, $a^2 - b^2 = (a+b)(a-b)$

$$\Rightarrow Z = 2 \times \lim_{x \to \frac{\pi}{4}} \frac{(1-\tan x)(1+\tan x)}{(1-\tan x)(1+\tan^2 x)} = 2 \lim_{x \to \frac{\pi}{4}} \frac{1+\tan x}{1+\tan^2 x}$$

Now put the value of x, we have-

$$\therefore Z = 2 \left(\frac{1+\tan\frac{\pi}{4}}{1+\tan^2\frac{\pi}{4}} \right) = 2 \times \left(\frac{2}{2}\right) = 2$$

Hence,

$$\lim_{x \to \frac{\pi}{4}} \frac{\mathrm{cosec}^2 x - 2}{\cot x - 1} = 2 \qquad \ldots.\text{ans}$$

3. Question

Evaluate the following limits:

$$\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\mathrm{cosec} x - 2}$$

Answer

As we need to find $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\mathrm{cosec} x - 2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc)

Let $Z = \lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\mathrm{cosec} x - 2} = \lim_{x \to \frac{\pi}{6}} \frac{\cot^2\frac{\pi}{6} - 3}{\mathrm{cosec}\frac{\pi}{6} - 2} = \frac{(\sqrt{3})^2 - 3}{2-2} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

As $Z = \lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\mathrm{cosec} x - 2}$

As, $a^2 - b^2 = (a+b)(a-b)$

$$\therefore Z = \lim_{x \to \frac{\pi}{6}} \frac{(\cot x - \sqrt{3})(\cot x + \sqrt{3})}{\mathrm{cosec} x - 2}$$

$$\Rightarrow Z = \lim_{x \to \frac{\pi}{6}} (\cot x + \sqrt{3}) \lim_{x \to \frac{\pi}{6}} \left(\frac{\cot x - \sqrt{3}}{\mathrm{cosec}\, x - 2} \right)$$

$$\Rightarrow Z = \left(\cot\frac{\pi}{6} + \sqrt{3}\right) \lim_{x \to \frac{\pi}{6}} \left(\frac{\cot x - \sqrt{3}}{\mathrm{cosec}\, x - 2} \right)$$

$$\Rightarrow Z = 2\sqrt{3} \lim_{x \to \frac{\pi}{6}} \left(\frac{\cot x - \sqrt{3}}{\mathrm{cosec}\, x - 2} \right)$$

Multiplying cosec x + 2 to both numerator and denominator-

$$Z = 2\sqrt{3}\lim_{x\to\frac{\pi}{6}}\left(\frac{\cot x-\sqrt{3}}{\operatorname{cosec} x-2}\right)\left(\frac{\operatorname{cosec} x+2}{\operatorname{cosec} x+2}\right) = 2\sqrt{3}\lim_{x\to\frac{\pi}{6}}\frac{(\cot x-\sqrt{3})(\operatorname{cosec} x+2)}{\operatorname{cosec}^2 x-4}$$

$$Z = 2\sqrt{3}\lim_{x\to\frac{\pi}{6}}(\operatorname{cosec} x+2)\times\lim_{x\to\frac{\pi}{6}}\frac{\cot x-\sqrt{3}}{\operatorname{cosec}^2 x-1-3}$$

As, $\operatorname{cosec}^2 x - 1 = \cot^2 x$

$$\therefore Z = 2\sqrt{3}\left(\operatorname{cosec}\frac{\pi}{6}+2\right)\times\lim_{x\to\frac{\pi}{6}}\frac{\cot x-\sqrt{3}}{\cot^2 x-3} = 8\sqrt{3}\times\lim_{x\to\frac{\pi}{6}}\frac{\cot x-\sqrt{3}}{(\cot x-\sqrt{3})(\cot x+\sqrt{3})}$$

$$\Rightarrow Z = 8\sqrt{3}\lim_{x\to\frac{\pi}{6}}\frac{1}{\cot x+\sqrt{3}} = 8\sqrt{3}\times\frac{1}{\cot\frac{\pi}{6}+\sqrt{3}} = \frac{8\sqrt{3}}{2\sqrt{3}} = 4$$

$$\therefore \lim_{x\to\frac{\pi}{6}}\frac{\cot^2 x-3}{\operatorname{cosec} x-2} = 4$$

4. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{4}}\frac{2-\operatorname{cosec}^2 x}{1-\cot x}$$

Answer

As we need to find $\lim_{x\to\frac{\pi}{4}}\frac{2-\operatorname{cosec}^2 x}{1-\cot x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

$$\text{Let } Z = \lim_{x\to\frac{\pi}{4}}\frac{2-\operatorname{cosec}^2 x}{1-\cot x} = \lim_{x\to\frac{\pi}{4}}\frac{2-\operatorname{cosec}^2\frac{\pi}{4}}{1-\cot\frac{\pi}{4}} = \frac{2-(\sqrt{2})^2}{1-1} = \frac{0}{0}\ (\text{indeterminate})$$

∴ we need to take steps to remove this form so that we can get a finite value.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

As $Z = \lim_{x\to\frac{\pi}{4}}\frac{2-\operatorname{cosec}^2 x}{1-\cot x}$

∵ $\operatorname{cosec}^2 x - 1 = \cot^2 x$

$$\therefore Z = \lim_{x\to\frac{\pi}{4}}\frac{1-(\operatorname{cosec}^2 x-1)}{1-\cot x} = \lim_{x\to\frac{\pi}{4}}\frac{1-\cot^2 x}{1-\cot x}$$

As, $a^2 - b^2 = (a+b)(a-b)$

Thus,

$$Z = \lim_{x\to\frac{\pi}{4}}\frac{(1-\cot x)(1+\cot x)}{1-\cot x} = \lim_{x\to\frac{\pi}{4}}(1+\cot x)$$

$$\therefore Z = 1+\cot\frac{\pi}{4} = 1+1 = 2$$

Hence,

$$\lim_{x\to\frac{\pi}{4}}\frac{2-\operatorname{cosec}^2 x}{1-\cot x} = 2 \qquad \ldots\text{ans}$$

5. Question

Evaluate the following limits:

$$\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$$

Answer

As we need to find $\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{x\to\pi}\frac{\sqrt{2+\cos\pi}-1}{(\pi-x)^2} = \frac{\sqrt{2-1}-1}{(\pi-\pi)^2} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

As $Z = \lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$

Multiplying numerator and denominator by √(2+cos x) + 1,we have-

$Z = \lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}\times\frac{\sqrt{2+\cos x}+1}{\sqrt{2+\cos x}+1}$

$\Rightarrow Z = \lim_{x\to\pi}\frac{\left(\sqrt{2+\cos x}\right)^2-1^2}{(\pi-x)^2\sqrt{2+\cos x}+1}$

{using $a^2 - b^2 = (a+b)(a-b)$}

$\Rightarrow Z = \lim_{x\to\pi}\frac{2+\cos x-1}{(\pi-x)^2}\lim_{x\to\pi}\frac{1}{\sqrt{2+\cos x}+1}$

{using basic algebra of limits}

$\Rightarrow Z = \frac{1}{\sqrt{2+\cos\pi}+1}\lim_{x\to\pi}\frac{1+\cos x}{(\pi-x)^2} = \frac{1}{2}\lim_{x\to\pi}\frac{1+\cos x}{(\pi-x)^2}$

As, $1+\cos x = 2\cos^2(x/2)$

$\therefore Z = \frac{1}{2}\lim_{x\to\pi}\frac{2\cos^2\left(\frac{x}{2}\right)}{(\pi-x)^2}$

Tip: Similar limit problems involving trigonometric ratios along with algebraic equations are mostly solved using sandwich theorem. $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$

So to solve this problem we need to have a sin term so that we can make use of sandwich theorem.

∵ sin(π/2 – x) = cos x

$\therefore Z = \frac{1}{2}\lim_{x\to\pi}\frac{2\sin^2\left(\frac{\pi}{2}-\frac{x}{2}\right)}{(\pi-x)^2}$

As x→π ⇒ π – x → 0

Let y = π – x

$Z = \frac{1}{2}\lim_{y\to 0}\frac{2\sin^2\left(\frac{y}{2}\right)}{y^2}$

To apply sandwich theorem we have to get the similar form as described below-

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$\therefore Z = \frac{1}{2}\lim_{y\to 0}\frac{2\sin^2\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)^2\times 4} = \frac{1}{4}\lim_{y\to 0}\left(\frac{\sin\left(\frac{y}{2}\right)}{\frac{y}{2}}\right)^2$

$\Rightarrow Z = \frac{1}{4}\times 1 = \frac{1}{4}$

Hence,

$$\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \frac{1}{4} \qquad \ldots.\text{ans}$$

6. Question

Evaluate the following limits:

$$\lim_{x\to\frac{3\pi}{2}}\frac{1+\operatorname{cosec}^2 x}{\cot^2 x}$$

Answer

As we need to find $\lim_{x\to\frac{3\pi}{2}}\frac{1+\operatorname{cosec}^2 x}{\cot^2 x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to\frac{3\pi}{2}}\frac{1+\operatorname{cosec}^2 x}{\cot^2 x} = \lim_{x\to\frac{3\pi}{2}}\frac{1+\operatorname{cosec}^2\left(\frac{3\pi}{2}\right)}{\cot^2\left(\frac{3\pi}{2}\right)} = \frac{1+1}{0} = \frac{2}{0} = \infty$

∴ Z is not taking an indeterminate form.

∴ Limiting the value of Z is not defined.

Hence,

$$\lim_{x\to\frac{3\pi}{2}}\frac{1+\operatorname{cosec}^2 x}{\cot^2 x} = \infty$$

Exercise 29.10

1. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{5^x-1}{\sqrt{4+x}-2}$$

Answer

As we need to find $\lim_{x\to 0}\frac{5^x-1}{\sqrt{4+x}-2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to 0}\frac{5^x-1}{\sqrt{4+x}-2} = \lim_{x\to 0}\frac{5^0-1}{\sqrt{4+0}-2} = \frac{1-1}{2-2} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

Note: While modifying be careful that you don't introduce any zero terms in the denominator

As Z = $\lim_{x\to 0}\frac{5^x-1}{\sqrt{4+x}-2}$

Multiplying both numerator and denominator by √(4+x)+2 so that we can remove the indeterminate form.

$\therefore$ Z = $\lim_{x\to 0}\frac{5^x-1}{\sqrt{4+x}-2}\times\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$

$\Rightarrow$ Z = $\lim_{x\to 0}\frac{(5^x-1)\sqrt{4+x}+2}{\left(\sqrt{4+x}\right)^2-2^2}$

{using $a^2 - b^2 = (a + b)(a - b)$}

$\Rightarrow$ Z = $\lim_{x\to 0}\frac{(5^x-1)\sqrt{4+x}+2}{4+x-4} = \lim_{x\to 0}\frac{(5^x-1)\sqrt{4+x}+2}{x}$

Using basic algebra of limits-

Z = $\lim_{x\to 0}\frac{(5^x-1)}{x}\times\lim_{x\to 0}\sqrt{4+x}+2 = \{\sqrt{4+0}+2\}\lim_{x\to 0}\frac{(5^x-1)}{x}$

$\Rightarrow$ Z = 4 $\lim_{x\to 0}\frac{(5^x-1)}{x}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

$\therefore$ Z = 4log 5

Or, $\lim_{x\to 0}\frac{5^x-1}{\sqrt{4+x}-2} = 4\log 5$

2. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\log(1+x)}{3^x-1}$$

Answer

As we need to find $\lim_{x\to 0}\frac{\log(1+x)}{3^x-1}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z $= \lim_{x\to 0}\frac{\log(1+x)}{3^x-1} = \lim_{x\to 0}\frac{\log(1+0)}{3^0-1} = \frac{\log 1}{1-1} = \frac{0}{0}$ (indeterminate form)

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

As Z = $\lim_{x\to 0}\frac{\log(1+x)}{3^x-1}$

To get the above forms, we need to divide numerator and denominator by x.

$\therefore$ Z = $\lim_{x\to 0}\frac{\frac{\log(1+x)}{x}}{\frac{3^x-1}{x}} = \frac{\lim_{x\to 0}\frac{\log(1+x)}{x}}{\lim_{x\to 0}\frac{3^x-1}{x}}$ {using basic limit algebra}

$\Rightarrow$ Z = $\frac{1}{\log 3}$ {using the formulae described above}

Hence,

$$\lim_{x\to0}\frac{\log(1+x)}{3^x-1}=\frac{1}{\log 3}$$

3. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{a^x+a^{-x}-2}{x^2}$$

Answer

As we need to find $\lim_{x\to0}\frac{a^x+a^{-x}-2}{x^2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to0}\frac{a^x+a^{-x}-2}{x^2}=\lim_{x\to0}\frac{a^0+a^{-0}-2}{x^2}=\frac{1+1-2}{0^2}=\frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

As Z = $\lim_{x\to0}\frac{a^x+a^{-x}-2}{x^2}=\lim_{x\to0}\frac{a^{-x}(a^{2x}-2a^x+1)}{x^2}$

∴ Z = $\lim_{x\to0}\frac{(a^{2x}-2a^x+1)}{a^x x^2}=\lim_{x\to0}\frac{(a^x-1)^2}{a^x x^2}$ {using $(a+b)^2 = a^2+b^2+2ab$}

Using algebra of limit, we can write that

Z = $\lim_{x\to0}\left(\frac{a^x-1}{x}\right)^2\times\lim_{x\to0}\frac{1}{a^x}$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$

∴ Z = $(\log a)^2\frac{1}{a^0}=(\log a)^2$

Hence,

$$\lim_{x\to0}\frac{a^x+a^{-x}-2}{x^2}=(\log a)^2$$

4. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{a^{mx}-1}{b^{nx}-1}, n\neq0$$

Answer

As we need to find $\lim_{x\to0}\frac{a^{mx}-1}{b^{nx}-1}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to0}\frac{a^{mx}-1}{b^{nx}-1}=\lim_{x\to0}\frac{a^{mo}-1}{b^{no}-1}=\frac{1-1}{1-1}=\frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get the desired forms, we need to include mx and nx as follows:

$$\therefore Z = \lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1} = \lim_{x\to 0}\frac{\frac{a^{mx}-1}{mx}\times mx}{\frac{b^{nx}-1}{nx}\times nx}$$

$$\Rightarrow Z = \frac{m}{n}\lim_{x\to 0}\frac{\frac{a^{mx}-1}{mx}}{\frac{b^{nx}-1}{nx}}$$

Using algebra of limits-

$$Z = \frac{m}{n}\frac{\lim_{x\to 0}\frac{a^{mx}-1}{mx}}{\lim_{x\to 0}\frac{b^{nx}-1}{nx}}$$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x}=\log a$

$$\therefore Z = \frac{m}{n}\frac{\log a}{\log b}, n \neq 0$$

Hence,

$$\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1} = \frac{m}{n}\frac{\log a}{\log b}, n \neq 0$$

5. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{a^x+b^x-2}{x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{a^x+b^x-2}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

$$\text{Let } Z = \lim_{x\to 0}\frac{a^x+b^x-2}{x} = \lim_{x\to 0}\frac{a^0+b^0-2}{x} = \frac{1+1-2}{0} = \frac{0}{0}\ (\text{indeterminate form})$$

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in a formula, we move as follows-

$$\text{As } Z = \lim_{x\to 0}\frac{a^x+b^x-2}{x}$$

$$\Rightarrow Z = \lim_{x\to 0}\frac{a^x-1+b^x-1}{x}$$

Using algebra of limits we have-

$Z = \lim_{x\to 0}\frac{a^x-1}{x} + \lim_{x\to 0}\frac{b^x-1}{x}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

$\therefore Z = \log a + \log b = \log ab$

Hence,

$$\lim_{x\to 0}\frac{a^x + b^x - 2}{x} = \log ab$$

6. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{9^x - 2.6^x + 4^x}{x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{9^x-2.6^x+4^x}{x^2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to 0}\frac{9^x-2.6^x+4^x}{x^2} = \lim_{x\to 0}\frac{9^0-2.6^0+4^0}{x^2} = \frac{1+1-2}{0} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

As $Z = \lim_{x\to 0}\frac{9^x-2.6^x+4^x}{x^2} = \lim_{x\to 0}\frac{(3^x)^2-2.3^x.2^x+(2^x)^2}{x^2}$

$\therefore Z = \lim_{x\to 0}\frac{(3^x-2^x)^2}{x^2}$

{using $(a-b)^2 = a^2+b^2-2ab$}

$Z = \lim_{x\to 0}\left(\frac{3^x-2^x}{x}\right)^2$

To apply the formula we need to bring the exact form present in the formula, so-

$Z = \lim_{x\to 0}\left(\frac{3^x-1-2^x+1}{x}\right)^2$

{Adding and subtracting 1 in numerator}

$\Rightarrow Z = \lim_{x\to 0}\left(\frac{3^x-1}{x} - \frac{2^x-1}{x}\right)^2$

Using algebra of limits-

$Z = \left(\lim_{x\to 0}\frac{3^x-1}{x} - \lim_{x\to 0}\frac{2^x-1}{x}\right)^2$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

$\therefore Z = (\log 3 - \log 2)^2 = \left(\log \frac{3}{2}\right)^2$

Hence,

$$\lim_{x \to 0} \frac{9^x - 2.6^x + 4^x}{x^2} = \left(\log \frac{3}{2}\right)^2$$

7. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{8^x - 4^x - 2^x + 1}{x^2}$$

Answer

As we need to find $\lim_{x \to 0} \frac{8^x - 4^x - 2^x + 1}{x^2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x \to 0} \frac{8^x - 4^x - 2^x + 1}{x^2} = \lim_{x \to 0} \frac{8^0 - 4^0 - 2^0 + 1}{x^2} = \frac{2-2}{0} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \to 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

As Z = $\lim_{x \to 0} \frac{8^x - 4^x - 2^x + 1}{x^2} = \lim_{x \to 0} \frac{4^x(2^x - 1) - 1(2^x - 1)}{x^2} = \lim_{x \to 0} \frac{(4^x - 1)(2^x - 1)}{x^2}$

Using Algebra of limits-

We have-

Z = $\lim_{x \to 0} \frac{(4^x - 1)}{x} \times \lim_{x \to 0} \frac{(2^x - 1)}{x}$

Use the formula: $\lim_{x \to 0} \frac{(a^x - 1)}{x} = \log a$

∴ Z = log 4 × log 2

∵ log 4 = log 2^2 = 2log 2

{using properties of log}

∴ Z = 2(log $2)^2$

Hence,

$$\lim_{x \to 0} \frac{8^x - 4^x - 2^x + 1}{x^2} = 2(\log 2)^2$$

8. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{a^{mx} - b^{nx}}{x}$$

Answer

As we need to find $\lim_{x \to 0} \frac{a^{mx} - b^{nx}}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z $= \lim\limits_{x\to 0}\frac{a^{mx}-b^{nx}}{x} = \lim\limits_{x\to 0}\frac{a^{mo}-b^{no}}{x} = \frac{1-1}{0} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim\limits_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get the desired forms, we need to include mx and nx as follows:

As Z $= \lim\limits_{x\to 0}\frac{a^{mx}-b^{nx}}{x}$

⇒ Z $= \lim\limits_{x\to 0}\frac{a^{mx}-1-b^{nx}+1}{x}$

{Adding and subtracting 1 in numerator}

⇒ Z $= \lim\limits_{x\to 0}\frac{a^{mx}-1}{x} - \lim\limits_{x\to 0}\frac{b^{nx}-1}{x}$

{using algebra of limits}

To get the form as present in the formula we multiply and divide m and n into both terms respectively:

∴ Z $= \lim\limits_{x\to 0}\frac{a^{mx}-1}{mx}\times m - \lim\limits_{x\to 0}\frac{b^{nx}-1}{nx}\times n$

Use the formula: $\lim\limits_{x\to 0}\frac{(a^x-1)}{x} = \log a$

∴ Z = m log a – n log b $= \log\left(\frac{a^m}{b^n}\right)$

{using properties of log}

Hence,

$$\lim_{x\to 0}\frac{a^{mx}-b^{nx}}{x} = \log\left(\frac{a^m}{b^n}\right)$$

9. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{a^x+b^x+c^x-3}{x}$$

Answer

As we need to find $\lim\limits_{x\to 0}\frac{a^x+b^x+c^x-3}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z $= \lim\limits_{x\to 0}\frac{a^x+b^x+c^x-3}{x} = \lim\limits_{x\to 0}\frac{a^0+b^0+c^0-3}{x} = \frac{1+1+1-3}{0} = \frac{0}{0}$

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim\limits_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in a formula, we move as follows-

As $Z = \lim_{x\to0}\frac{a^x+b^x+c^x-3}{x}$

$\Rightarrow Z = \lim_{x\to0}\frac{a^x-1+b^x-1+c^x-1}{x}$

Using algebra of limits we have-

$Z = \lim_{x\to0}\frac{a^x-1}{x} + \lim_{x\to0}\frac{b^x-1}{x} + \lim_{x\to0}\frac{c^x-1}{x}$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$

$\therefore Z = \log a + \log b + \log c = \log abc$

Hence,

$$\lim_{x\to0}\frac{a^x + b^x + c^x - 3}{x} = \log abc$$

10. Question

Evaluate the following limits:

$$\lim_{x\to2}\frac{x-2}{\log_a(x-1)}$$

Answer

As we need to find $\lim_{x\to2}\frac{x-2}{\log_a(x-1)}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to2}\frac{x-2}{\log_a(x-1)} = \lim_{x\to2}\frac{2-2}{\log_a(2-1)} = \frac{2-2}{\log 1} = \frac{0}{0}$ (indeterminate form)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in a formula, we move as follows-

$$Z = \lim_{x\to2}\frac{x-2}{\log_a(1+x-2)}$$

As x→2 ∴ x-2 →0

Let x-2 = y

$\therefore Z = \lim_{y\to0}\frac{y}{\log_a(1+y)} = \lim_{y\to0}\frac{1}{\frac{\log_a(1+y)}{y}}$

We can’t use the formula directly as the base of log is we need to change this to e.

Applying the formula for change of base-

We have- $\log_a(1+y) = \frac{\log_e(1+y)}{\log_e a}$

$\therefore Z = \lim_{y\to0}\frac{1}{\frac{\frac{\log_e(1+y)}{\log_e a}}{y}} = \frac{\log_e a}{\lim_{y\to0}\frac{\log_e(1+y)}{y}}$

Use the formula: $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

$\therefore$ Z = $\log_e a = \log a$

Hence,

$$\lim_{x\to2}\frac{x-2}{\log_a(x-1)}=\log a$$

11. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{5^x+3^x+2^x-3}{x}$$

Answer

As we need to find $\lim_{x\to0}\frac{5^x+3^x+2^x-3}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z $=\lim_{x\to0}\frac{5^x+3^x+2^x-3}{x}=\lim_{x\to0}\frac{5^0+3^0+2^0-3}{x}=\frac{1+1+1-3}{0}=\frac{0}{0}$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in formula, we move as follows-

As Z $=\lim_{x\to0}\frac{5^x+3^x+2^x-3}{x}$

$\Rightarrow$ Z $=\lim_{x\to0}\frac{5^x-1+3^x-1+2^x-1}{x}$

Using algebra of limits we have-

Z $=\lim_{x\to0}\frac{5^x-1}{x}+\lim_{x\to0}\frac{3^x-1}{x}+\lim_{x\to0}\frac{2^x-1}{x}$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$

$\therefore Z=\log 5+\log 3+\log 2=\log(5\times3\times2)$

Hence,

$$\lim_{x\to0}\frac{5^x+3^x+2^x-3}{x}=\log 30$$

12. Question

Evaluate the following limits:

$$\lim_{x\to\infty}(a^{1/x}-1)x$$

Answer

As we need to find $\lim_{x\to\infty}(a^{1/x}-1)x$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to\infty}\left(a^{\frac{1}{x}}-1\right)x = \lim_{x\to\infty}\left(a^{\frac{1}{\infty}}-1\right)\times\infty = 0\times\infty = \text{(indeterminate)}$

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in formula, we move as follows-

Let 1/x = y

As x→∞ ⇒ y→ 0

∴ Z can be rewritten as-

Z = $\lim_{y\to 0}\frac{(a^y-1)}{y}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

∴ Z = $\log a$

Hence,

$\lim_{x\to\infty}\left(a^{\frac{1}{x}}-1\right)x = \log a$

13. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{a^{mx}-b^{nx}}{\sin kx}$$

Answer

As we need to find $\lim_{x\to 0}\frac{a^{mx}-b^{nx}}{\sin kx}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to 0}\frac{a^{mx}-b^{nx}}{\sin kx} = \lim_{x\to 0}\frac{a^{m0}-b^{n0}}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}\ \text{(indeterminate form)}$

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits and also use of sandwich theorem - $\lim_{x\to 0}\frac{\sin x}{x} = 1$

To get the desired forms, we need to include mx and nx as follows:

As Z = $\lim_{x\to 0}\frac{a^{mx}-b^{nx}}{\sin kx}$

⇒ Z = $\lim_{x\to 0}\frac{a^{mx}-1-b^{nx}+1}{\sin kx}$ {Adding and subtracting 1 in numerator}

$\Rightarrow Z = \lim_{x\to 0}\frac{a^{mx}-1}{\sin kx} - \lim_{x\to 0}\frac{b^{nx}-1}{\sin kx}$

{using algebra of limits}

To get the form as present in the formula we multiply and divide x into both terms respectively:

$\therefore Z = \lim_{x\to 0}\frac{\frac{a^{mx}-1}{x}}{\frac{(\sin kx)}{x}} - \lim_{x\to 0}\frac{\frac{b^{nx}-1}{x}}{\frac{(\sin kx)}{x}}$

{manipulating to get the forms present in formulae}

$Z = \lim_{x\to 0}\frac{\frac{a^{mx}-1}{mx}\times m}{\frac{(\sin kx)}{kx}\times k} - \lim_{x\to 0}\frac{\frac{b^{nx}-1}{nx}\times n}{\frac{(\sin kx)}{kx}\times k}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\therefore Z = \frac{m\log a}{k} - \frac{n\log b}{k} = \frac{1}{k}(m\log a - n\log b)$

Hence,

$$\lim_{x\to 0}\frac{a^{mx}-b^{nx}}{\sin kx} = \frac{1}{k}\log\left(\frac{a^m}{b^n}\right)$$

14. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{a^x+b^x-c^c-d^x}{x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{a^x+b^x-c^x-d^x}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to 0}\frac{a^x+b^x-c^x-d^x}{x} = \lim_{x\to 0}\frac{a^0+b^0-c^0-d^0}{x} = \frac{1+1-1-1}{0} = \frac{0}{0}$

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in a formula, we move as follows-

As $Z = \lim_{x\to 0}\frac{a^x+b^x-c^x-d^x}{x}$

$\Rightarrow Z = \lim_{x\to 0}\frac{a^x-1+b^x-1-c^x+1-d^x+1}{x}$

Using algebra of limits we have-

$Z = \lim_{x\to 0}\frac{a^x-1}{x} + \lim_{x\to 0}\frac{b^x-1}{x} - \lim_{x\to 0}\frac{c^x-1}{x} - \lim_{x\to 0}\frac{d^x-1}{x}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

$\therefore Z = \log a + \log b - \log c - \log d = \log\frac{ab}{cd}$

Hence,

$$\lim_{x\to0}\frac{a^x+b^x-c^x-d^x}{x}=\log\left(\frac{ab}{cd}\right)$$

15. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{e^x-1+\sin x}{x}$$

Answer

As we need to find $\lim_{x\to0}\frac{e^x-1+\sin x}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to0}\frac{e^x-1+\sin x}{x}=\lim_{x\to0}\frac{e^0-1+\sin 0}{0}=\frac{1-1}{0}=\frac{0}{0}$ (indeterminate form)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x\to0}\frac{\sin x}{x}=1$

As Z = $\lim_{x\to0}\frac{e^x-1+\sin x}{x}$

⇒ Z = $\lim_{x\to0}\frac{e^x-1}{x}+\lim_{x\to0}\frac{\sin x}{x}$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to0}\frac{\sin x}{x}=1$

∴ Z = log e + 1

{∵ log e = 1}

⇒ Z = 1+1 = 2

Hence,

$$\lim_{x\to0}\frac{e^x-1+\sin x}{x}=2$$

16. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{\sin 2x}{e^x-1}$$

Answer

As we need to find $\lim_{x\to0}\frac{\sin 2x}{e^x-1}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to0}\frac{\sin 2x}{e^x-1}=\lim_{x\to0}\frac{\sin 0}{e^0-1}=\frac{0}{1-1}=\frac{0}{0}$ (indeterminate form)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula$\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x\to0}\frac{\sin x}{x} = 1$

As Z = $\lim_{x\to0}\frac{\sin 2x}{e^x-1}$

To get the desired form to apply the formula we need to divide numerator and denominator by x.

⇒ Z = $\lim_{x\to0}\frac{\frac{\sin 2x}{x}}{\frac{e^x-1}{x}}$

Using algebra of limits, we have-

Z = $\frac{\lim_{x\to0}\frac{\sin 2x}{2x}\times2}{\lim_{x\to0}\frac{e^x-1}{x}}$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\sin x}{x} = 1$

∴ Z = $\frac{2}{\log e}$

{∵ log e = 1}

⇒ Z = 2

Hence,

$$\lim_{x\to0}\frac{\sin 2x}{e^x-1} = 2$$

17. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{e^{\sin x}-1}{x}$$

Answer

As we need to find $\lim_{x\to0}\frac{e^{\sin x}-1}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to0}\frac{e^{\sin x}-1}{x} = \lim_{x\to0}\frac{e^{\sin o}-1}{0} = \frac{1-1}{0} = \frac{0}{0}$ (indeterminate form)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula$\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x\to0}\frac{\sin x}{x} = 1$

As Z = $\lim_{x\to0}\frac{e^{\sin x}-1}{x}$

To get rid of indeterminate form we will divide numerator and denominator by sin x

$\therefore Z = \lim_{x \to 0} \frac{\frac{e^{\sin x}-1}{\frac{\sin x}{x}}}{\sin x}$

Using Algebra of limits we have-

$Z = \frac{\lim_{x \to 0} \frac{e^{\sin x}-1}{\frac{\sin x}{x}}}{\lim_{x \to 0} \sin x} = \frac{A}{B}$

Where, $A = \lim_{x \to 0} \frac{e^{\sin x}-1}{\sin x}$

and $B = \lim_{x \to 0} \frac{x}{\sin x} = 1$

{from sandwich theorem}

As $A = \lim_{x \to 0} \frac{e^{\sin x}-1}{\sin x}$

Let, sin x =y

As x→0 ⇒ y→0

$\therefore A = \lim_{y \to 0} \frac{e^{y}-1}{y}$

Using $\lim_{x \to 0} \frac{(a^x-1)}{x} = \log a$

A = log e = 1

$\therefore Z = \frac{A}{B} = \frac{1}{1} = 1$

Hence,

$$\lim_{x \to 0} \frac{e^{\sin x} - 1}{x} = 1$$

18. Question

Evaluate the following limits:

$$\lim_{x \to 0} \frac{e^{2x} - e^{x}}{\sin 2x}$$

Answer

As we need to find $\lim_{x \to 0} \frac{e^{2x}-e^{x}}{\sin 2x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x \to 0} \frac{e^{2x}-e^{x}}{\sin 2x} = \lim_{x \to 0} \frac{e^{0}-e^{0}}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}$ (indeterminate form)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \to 0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

As $Z = \lim_{x \to 0} \frac{e^{2x}-e^{x}}{\sin 2x}$

Adding and subtracting 1 in the numerator to get the desired form

$\Rightarrow Z = \lim_{x\to 0} \frac{e^{2x}-1-e^{x}+1}{\sin 2x}$

$\Rightarrow Z = \lim_{x\to 0} \frac{e^{2x}-1}{\sin 2x} - \lim_{x\to 0} \frac{e^{x}-1}{\sin 2x}$

{using algebra of limits}

To get the desired form to apply the formula we need to divide numerator and denominator by x.

$\Rightarrow Z = \lim_{x\to 0} \frac{\frac{e^{2x}-1}{2x}}{\frac{\sin 2x}{2x}} - \lim_{x\to 0} \frac{\frac{e^{x}-1}{x}}{\frac{\sin 2x}{2x}\times 2}$

Using algebra of limits, we have-

$Z = \frac{\lim_{x\to 0}\frac{e^{2x}-1}{2x}}{\lim_{x\to 0}\frac{\sin 2x}{2x}} - \frac{\lim_{x\to 0}\frac{e^{x}-1}{x}}{\lim_{x\to 0}\frac{\sin 2x}{2x}\times 2}$

Use the formula: $\lim_{x\to 0} \frac{(a^{x}-1)}{x} = \log a$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\therefore Z = \frac{\log e}{1} - \frac{\log e}{2} = 1 - \frac{1}{2} = \frac{1}{2}$

{$\because$ log e = 1}

$\Rightarrow Z = 1/2$

Hence,

$\lim_{x\to 0} \frac{e^{2x}-e^{x}}{\sin 2x} = \frac{1}{2}$

19. Question

Evaluate the following limits:

$\lim_{x\to a} \frac{\log x - \log a}{x - a}$

Answer

As we need to find $\lim_{x\to a} \frac{\log x - \log a}{x-a}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to a} \frac{\log x-\log a}{x-a} = \lim_{x\to a} \frac{\log a-\log a}{a-a} = \frac{0}{0}$ (indeterminate form)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0} \frac{(a^{x}-1)}{x} = \log a$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

As $Z = \lim_{x\to a} \frac{\log x-\log a}{x-a}$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

∴ We proceed as follows-

$Z = \lim_{x\to a} \frac{\log x-\log a}{x-a} = \lim_{x\to a} \frac{\log\left(\frac{x}{a}\right)}{x-a}$

$\Rightarrow Z = \lim_{x\to a} \frac{\log\left(\frac{x}{a}\right)}{a\left(\frac{x}{a}-1\right)}$

$\Rightarrow Z = \lim_{x\to a}\frac{\log\left(1+\frac{x}{a}-1\right)}{a\left(\frac{x}{a}-1\right)}$

∵ x→a ⇒ x/a →1

⇒ x/a – 1 → 0

Let, (x/a)-1 = y

∴ y→0

Hence, Z can be rewritten as-

$Z = \lim_{y\to 0}\frac{\log(1+y)}{a(y)}$

Use the formula: $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

$\therefore Z = \frac{1}{a}\lim_{y\to 0}\frac{\log(1+y)}{(y)} = \frac{1}{a}$

Hence,

$$\lim_{x\to a}\frac{\log x - \log a}{x - a} = \frac{1}{a}$$

20. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\log(a+x)-\log(a-x)}{x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{\log(a+x)-\log(a-x)}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to 0}\frac{\log(a+x)-\log(a-x)}{x} = \lim_{x\to 0}\frac{\log a-\log a}{0} = \frac{0}{0}$ (indeterminate)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

As $Z = \lim_{x\to 0}\frac{\log(a+x)-\log(a-x)}{x}$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

∴ We proceed as follows-

$Z = \lim_{x\to 0}\frac{\log(a+x)-\log(a-x)}{x} = \lim_{x\to 0}\frac{\log(\frac{a+x}{a-x})}{x}$

$\Rightarrow Z = \lim_{x\to 0}\frac{\log(\frac{a+x}{a-x})}{x}$

$\Rightarrow Z = \lim_{x\to 0}\frac{\log\left(1+\frac{2x}{a-x}\right)}{x}$

To apply the formula of logarithmic limit we need $\frac{2x}{a-x}$ denominator

∴ multiplying $\frac{2}{a-x}$ in numerator and denominator

Hence, Z can be rewritten as-

$Z = \lim_{x\to 0} \frac{\log\left(1+\frac{2x}{a-x}\right)}{\frac{2x}{a-x}} \times \frac{2}{a-x}$

$\Rightarrow Z = \lim_{x\to 0} \frac{\log\left(1+\frac{2x}{a-x}\right)}{\frac{2x}{a-x}} \times \lim_{x\to 0} \frac{2}{a-x}$

{Using algebra of limits}

$\Rightarrow Z = \frac{2}{a} \lim_{x\to 0} \frac{\log\left(1+\frac{2x}{a-x}\right)}{\frac{2x}{a-x}}$

As, x→0 ⇒ $\frac{2x}{a-x} \to 0$

Let, $\frac{2x}{a-x} = y$

∴ $Z = \frac{2}{a} \lim_{y\to 0} \frac{\log(1+y)}{y}$

Use the formula: $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

∴ $Z = \frac{2}{a} \lim_{y\to 0} \frac{\log(1+y)}{(y)} = \frac{2}{a}$

Hence,

$$\lim_{x\to 0} \frac{\log(a+x) - \log(a-x)}{x} = \frac{2}{a}$$

21. Question

Evaluate the following limits:

$$\lim_{x\to 0} \frac{\log(2+x) + \log 0.5}{x}$$

Answer

As we need to find $\lim_{x\to 0} \frac{\log(2+x)+\log 0.5}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to 0} \frac{\log(2+x)+\log 0.5}{x} = \frac{\log(2+0)+\log 0.5}{0} = \frac{0}{0}$ (indeterminate)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

As $Z = \lim_{x\to 0} \frac{\log(2+x)+\log 0.5}{x}$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

∴ We proceed as follows-

$Z = \lim_{x\to 0} \frac{\log(2+x)+\log 0.5}{x} = \lim_{x\to 0} \frac{\log\{(2+x)\times 0.5\}}{x}$

{using properties of log}

$\Rightarrow$ Z = $\lim_{x\to0}\frac{\log\left(1+\frac{x}{2}\right)}{x}$

To apply the formula of logarithmic limit, we need the x/2 denominator

∴ multiplying 1/2 in numerator and denominator

Hence, Z can be rewritten as-

Z = $\lim_{x\to0}\frac{\log\left(1+\frac{x}{2}\right)}{\frac{x}{2}}\times\frac{1}{2}$

$\Rightarrow$ Z = $\frac{1}{2}\lim_{x\to0}\frac{\log\left(1+\frac{x}{2}\right)}{\frac{x}{2}}$

{Using algebra of limits}

As x→0 $\Rightarrow \frac{x}{2}\to0$

Let, $\frac{x}{2}=y$

∴ Z = $\frac{1}{2}\lim_{y\to0}\frac{\log(1+y)}{y}$

Use the formula: $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

∴ Z = $\frac{1}{2}\lim_{y\to0}\frac{\log(1+y)}{(y)}=\frac{1}{2}$

Hence,

$$\lim_{x\to0}\frac{\log(2+x)+\log0.5}{x}=\frac{1}{2}$$

22. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{\log(a+x)-\log a}{x}$$

Answer

As we need to find $\lim_{x\to0}\frac{\log(a+x)-\log a}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to0}\frac{\log(a+x)-\log a}{x}=\lim_{x\to0}\frac{\log a-\log a}{0}=\frac{0}{0}$(indeterminate)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula$\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

As Z = $\lim_{x\to0}\frac{\log(a+x)-\log(a)}{x}$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

∴ We proceed as follows-

Z = $\lim_{x\to 0}\frac{\log(a+x)-\log(a)}{x} = \lim_{x\to 0}\frac{\log(\frac{a+x}{a})}{x}$ {using properties of log}

⇒ Z = $\lim_{x\to 0}\frac{\log(1+\frac{x}{a})}{x}$

To apply the formula of logarithmic limit, we need x/a in the denominator

∴ multiplying 1/a in numerator and denominator

Hence, Z can be rewritten as-

Z = $\lim_{x\to 0}\frac{\log\left(1+\frac{x}{a}\right)}{\frac{x}{a}} \times \frac{1}{a}$

⇒ Z = $\frac{1}{a}\lim_{x\to 0}\frac{\log\left(1+\frac{x}{a}\right)}{\frac{x}{a}}$

{Using algebra of limits}

As x→0 ⇒ $\frac{x}{a} \to 0$

Let, $\frac{x}{a} = y$

∴ Z = $\frac{1}{a}\lim_{y\to 0}\frac{\log(1+y)}{y}$

Use the formula: $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

∴ Z = $\frac{1}{a}\lim_{y\to 0}\frac{\log(1+y)}{(y)} = \frac{1}{a}$

Hence,

$$\lim_{x\to 0}\frac{\log(a+x)-\log(a)}{x} = \frac{1}{a}$$

23. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x} = \lim_{x\to 0}\frac{\log 3-\log 3}{0} = \frac{0}{0}$(indeterminate)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula$\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

As Z = $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

∴ We proceed as follows-

$Z = \lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x} = \lim_{x\to 0}\frac{\log(\frac{3+x}{3-x})}{x}$

$\Rightarrow Z = \lim_{x\to 0}\frac{\log(\frac{3+x}{3-x})}{x}$

$\Rightarrow Z = \lim_{x\to 0}\frac{\log\left(1+\frac{2x}{3-x}\right)}{x}$

To apply the formula of logarithmic limit we need $\frac{2x}{3-x}$ denominator

∴ multiplying $\frac{2}{3-x}$ in numerator and denominator

Hence, Z can be rewritten as-

$Z = \lim_{x\to 0}\frac{\log\left(1+\frac{2x}{3-x}\right)}{\frac{2x}{3-x}} \times \frac{2}{3-x}$

$\Rightarrow Z = \lim_{x\to 0}\frac{\log\left(1+\frac{2x}{3-x}\right)}{\frac{2x}{3-x}} \times \lim_{x\to 0}\frac{2}{3-x}$

{Using algebra of limits}

$\Rightarrow Z = \frac{2}{3}\lim_{x\to 0}\frac{\log\left(1+\frac{2x}{3-x}\right)}{\frac{2x}{3-x}}$

As, x→0 ⇒ $\frac{2x}{3-x} \to 0$

Let, $\frac{2x}{3-x} = y$

∴ $Z = \frac{2}{3}\lim_{y\to 0}\frac{\log(1+y)}{y}$

Use the formula: $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

∴ $Z = \frac{2}{3}\lim_{y\to 0}\frac{\log(1+y)}{(y)} = \frac{2}{3}$

Hence,

$$\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x} = \frac{2}{3}$$

24. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{8^x-2^x}{x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{8^x-2^x}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to 0}\frac{8^x-2^x}{x} = \lim_{x\to 0}\frac{8^0-2^0}{0} = \frac{1-1}{0} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

and $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

This question is a direct application of limits formula of exponential and logarithmic limits.

As Z = $\lim_{x\to0}\frac{8^x-2^x}{x}$

⇒ Z = $\lim_{x\to0}\frac{8^x-1-2^x+1}{x}$

{Adding and subtracting 1 in numerator}

⇒ Z = $\lim_{x\to0}\frac{8^x-1}{x}-\lim_{x\to0}\frac{2^x-1}{x}$

{using algebra of limits}

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$

∴ Z = log 8 – log 2 = $\log\left(\frac{8}{2}\right)=\log 4$

{using properties of log}

Hence,

$$\lim_{x\to0}\frac{8^x-2^x}{x}=\log 4$$

25. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{x(2^x-1)}{1-\cos x}$$

Answer

As we need to find $\lim_{x\to0}\frac{x(2^x-1)}{1-\cos x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to0}\frac{x(2^x-1)}{1-\cos x}=\lim_{x\to0}\frac{0(2^0-1)}{1-\cos 0}=\frac{0}{1-1}=\frac{0}{0}$ (indeterminate form)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x\to0}\frac{\sin x}{x}=1$

As Z = $\lim_{x\to0}\frac{x(2^x-1)}{1-\cos x}$

As, 1-cos x = $2\sin^2(x/2)$

∴ Z = $\lim_{x\to0}\frac{x(2^x-1)}{2\sin^2\left(\frac{x}{2}\right)}$

⇒ Z = $\frac{1}{2}\lim_{x\to0}\frac{x(2^x-1)}{\sin^2\left(\frac{x}{2}\right)}$

To get the desired form to apply the formula we need to divide numerator and denominator by x^2.

$\Rightarrow Z = \frac{1}{2}\lim_{x\to0}\frac{\frac{x(2^x-1)}{x^2}}{\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2\times4}} = \frac{4}{2}\lim_{x\to0}\frac{\frac{(2^x-1)}{x}}{\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2}$

Using algebra of limits, we have-

$Z = 2\frac{\lim_{x\to0}\frac{(2^x-1)}{x}}{\lim_{x\to0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2}$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\sin x}{x} = 1$

$\therefore Z = 2\,\frac{\log 2}{1^2}$

$\Rightarrow Z = 2\log 2$

Hence,

$$\lim_{x\to0}\frac{x(2^x-1)}{1-\cos x} = 2\log 2$$

26. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{\sqrt{1+x}-1}{\log(1+x)}$$

Answer

As we need to find $\lim_{x\to0}\frac{\sqrt{1+x}-1}{\log(1+x)}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to0}\frac{\sqrt{1+x}-1}{\log(1+x)} = \lim_{x\to0}\frac{\sqrt{1+0}-1}{\log(1+0)} = \frac{1-1}{0} = \frac{0}{0}$ (indeterminate)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x} = 1$

As $Z = \lim_{x\to0}\frac{\sqrt{1+x}-1}{\log(1+x)}$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

∴ multiplying numerator and denominator by $\sqrt{1+x}+1$

$\Rightarrow Z = \lim_{x\to0}\frac{\sqrt{1+x}-1}{\log(1+x)}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$\Rightarrow Z = \lim_{x\to0}\frac{\left(\sqrt{1+x}\right)^2-1^2}{\log(1+x)\times(\sqrt{1+x}+1)}$

{using $(a+b)(a-b)=a^2-b^2$}

$\Rightarrow Z = \lim_{x\to0}\frac{1+x-1}{\log(1+x)}\times\lim_{x\to0}\frac{1}{\sqrt{1+x}+1}$

$\Rightarrow Z = \lim_{x\to0}\frac{x}{\log(1+x)}\times\frac{1}{\sqrt{1+0}+1} = \frac{1}{2}\lim_{x\to0}\frac{x}{\log(1+x)}$

Use the formula: $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

∴ Z = 1/2

Hence,

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{\log(1+x)} = \frac{1}{2}$$

27. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{\log|1+x^3|}{\sin^3 x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{\log|1+x^3|}{\sin^3 x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to 0}\frac{\log|1+x^3|}{\sin^3 x} = \lim_{x\to 0}\frac{\log|1+0^3|}{\sin^3 0} = \frac{\log 1}{0} = \frac{0}{0}$ (indeterminate)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x\to 0}\frac{\sin x}{x} = 1$

As Z = $\lim_{x\to 0}\frac{\log|1+x^3|}{\sin^3 x}$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

∴ dividing numerator and denominator by x^3

⇒ Z = $\lim_{x\to 0}\frac{\frac{\log|1+x^3|}{x^3}}{\frac{\sin^3 x}{x^3}}$

⇒ Z = $\lim_{x\to 0}\frac{\frac{\log|1+x^3|}{x^3}}{\left(\frac{\sin x}{x}\right)^3}$

⇒ Z = $\frac{\lim_{x\to 0}\frac{\log|1+x^3|}{x^3}}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3}$

{using algebra of limits}

Use the formula: $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$

∴ Z = 1/1

Hence,

$$\lim_{x\to 0}\frac{\log|1+x^3|}{\sin^3 x} = 1$$

28. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$$

Answer

As we need to find $\lim_{x\to\frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to\frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} = \frac{a^{\cot\frac{\pi}{2}} - a^{\cos\frac{\pi}{2}}}{\cot\frac{\pi}{2} - \cos\frac{\pi}{2}} = \frac{1-1}{0} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

As Z = $\lim_{x\to\frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$

⇒ Z = $\lim_{x\to\frac{\pi}{2}} \frac{a^{\cos x}\left(\frac{a^{\cot x}}{a^{\cos x}} - 1\right)}{\cot x - \cos x}$

⇒ Z = $\lim_{x\to\frac{\pi}{2}} \frac{a^{\cos x}\left(a^{(\cot x - \cos x)} - 1\right)}{\cot x - \cos x}$

{using properties of exponents}

⇒ Z = $\lim_{x\to\frac{\pi}{2}} \frac{\left(a^{(\cot x - \cos x)} - 1\right)}{\cot x - \cos x} \times \lim_{x\to\frac{\pi}{2}} a^{\cos x}$

{using algebra of limits}

⇒ Z = $\lim_{x\to\frac{\pi}{2}} \frac{\left(a^{(\cot x - \cos x)} - 1\right)}{\cot x - \cos x} \times a^{\cos\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}} \frac{\left(a^{(\cot x - \cos x)} - 1\right)}{\cot x - \cos x} \times a^0$

∴ Z = $\lim_{x\to\frac{\pi}{2}} \frac{\left(a^{(\cot x - \cos x)} - 1\right)}{\cot x - \cos x}$

As, x→ (π/2)

∴ cot(π/2) – cos(π/2) → 0

Let, y = cot x – cos x

∴ if x→π/2 ⇒ y→0

Hence, Z can be rewritten as-

Z = $\lim_{y\to 0} \frac{(a^y - 1)}{y}$

Use the formula: $\lim_{x\to 0} \frac{(a^x - 1)}{x} = \log a$

∴ Z = log a

Hence,

$$\lim_{x\to\frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} = \log a$$

29. Question

Evaluate the following limits:

$$\lim_{x\to 0} \frac{e^x - 1}{\sqrt{1-\cos x}}$$

Answer

As we need to find $\lim_{x\to 0} \frac{e^x-1}{\sqrt{1-\cos x}}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to 0} \frac{e^x-1}{\sqrt{1-\cos x}} = \frac{e^0-1}{\sqrt{1-\cos 0}} = \frac{1-1}{\sqrt{1-1}} = \frac{0}{0}$(indeterminate)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x\to 0} \frac{\sin x}{x} = 1$

As Z = $\lim_{x\to 0} \frac{e^x-1}{\sqrt{1-\cos x}}$

To apply the formula we need to get the form as present in the formula. So we proceed as follows-

∵ Z = $\lim_{x\to 0} \frac{e^x-1}{\sqrt{1-\cos x}}$

Multiplying numerator and denominator by √(1+cos x)

⇒ Z = $\lim_{x\to 0} \frac{e^x-1}{\sqrt{1-\cos x}} \times \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}$

Using (a+b)(a-b) = a^2-b^2

Z = $\lim_{x\to 0} \frac{(e^x-1)\sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}}$

∵ $\sqrt{(1-\cos^2 x)}$ = sin x

⇒ Z = $\lim_{x\to 0} \frac{(e^x-1)}{\sin x} \times \lim_{x\to 0} \sqrt{1+\cos x}$

{using algebra of limits}

⇒ Z = $\lim_{x\to 0} \frac{(e^x-1)}{\sin x} \times \sqrt{1+\cos 0} = \sqrt{2} \lim_{x\to 0} \frac{(e^x-1)}{\sin x}$

Dividing numerator and denominator by x-

Z = $\sqrt{2} \lim_{x\to 0} \frac{\left(\frac{e^x-1}{x}\right)}{\frac{\sin x}{x}}$

⇒ Z = $\sqrt{2}\, \frac{\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)}{\lim_{x\to 0} \frac{\sin x}{x}}$

Use the formula: $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\therefore Z = \sqrt{2}\ \frac{\log e}{1}$

$\{\because \log e = 1\}$

Hence,

$$\lim_{x\to 0}\frac{e^x - 1}{\sqrt{1-\cos x}} = \sqrt{2}$$

30. Question

Evaluate the following limits:

$$\lim_{x\to 5}\frac{e^x - e^5}{x-5}$$

Answer

As we need to find $\lim_{x\to 5}\frac{e^x - e^5}{x-5}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \frac{\lim_{x\to 5}(e^x - e^5)}{x-5} = \frac{(e^5-e^5)}{5-5} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

As $Z = \lim_{x\to 5}\frac{e^x - e^5}{x-5}$

$\Rightarrow Z = \lim_{x\to 5}\frac{e^5(\frac{e^x}{e^5}-1)}{x-5}$

$\Rightarrow Z = \lim_{x\to 5}\frac{e^5(e^{x-5}-1)}{x-5}$

{using properties of exponents}

$\Rightarrow Z = e^5\lim_{x\to 5}\frac{(e^{x-5}-1)}{x-5}$

{using algebra of limits}

As, x→ 5

∴ x-5→ 0

Let, y = x-5

∴ if x→5 ⇒ y→0

Hence, Z can be rewritten as-

$Z = e^5\lim_{y\to 0}\frac{(e^y-1)}{y}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

$\therefore Z = e^5 \log e$

{∵ log e = 1}

Hence,

$$\lim_{x\to5}\frac{e^x - e^5}{x-5} = e^5$$

31. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{e^{x+2} - e^2}{x}$$

Answer

As we need to find $\lim_{x\to0}\frac{e^{x+2}-e^2}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\frac{\lim_{x\to0}(e^{x+2}-e^2)}{x} = \frac{(e^2-e^2)}{0} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

As Z = $\lim_{x\to0}\frac{e^{x+2}-e^2}{x}$

⇒ Z = $\lim_{x\to0}\frac{e^2(e^x-1)}{x}$

⇒ Z = $e^2\lim_{x\to0}\frac{(e^x-1)}{x}$

{using algebra of limits}

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$

∴ Z = e^2 log e

{∵ log e = 1}

Hence,

$$\lim_{x\to0}\frac{e^{x+2} - e^2}{x} = e^2$$

32. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{2}}\frac{e^{\cos x} - 1}{\cos x}$$

Answer

As we need to find $\lim_{x\to\frac{\pi}{2}}\frac{e^{\cos x}-1}{\cos x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to\frac{\pi}{2}} \frac{e^{\cos x}-1}{\cos x} = \frac{e^{\cos\frac{\pi}{2}}-1}{\cos\frac{\pi}{2}} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

As x→ π/2

∴ cos x → 0

Let, y = cos x

∴ if x→ π/2 ⇒ y→0

Hence, Z can be rewritten as-

$$\lim_{y\to 0} \frac{(e^y-1)}{y}$$

Use the formula: $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$

∴ Z = 1

{∵ log e = 1}

Hence,

$$\lim_{x\to\frac{\pi}{2}} \frac{e^{\cos x}-1}{\cos x} = 1$$

33. Question

Evaluate the following limits:

$$\lim_{x\to 0} \frac{e^{3+x}-\sin x-e^3}{x}$$

Answer

As we need to find $\lim_{x\to 0} \frac{e^{3+x}-\sin x-e^3}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to 0} \frac{e^{3+x}-\sin x-e^3}{x} = \frac{e^{3+0}-\sin 0-e^3}{0} = \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

As Z = $\lim_{x\to 0} \frac{e^{3+x}-\sin x-e^3}{x}$

$\Rightarrow Z = \lim_{x\to5} \frac{e^3(e^x-1)-\sin x}{x}$

$\Rightarrow Z = e^3 \lim_{x\to5} \frac{(e^x-1)}{x} - \lim_{x\to0} \frac{\sin x}{x}$

{using algebra of limits}

Use the formula: $\lim_{x\to0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0} \frac{\sin x}{x} = 1$

$\therefore Z = e^3 \log e - 1$ {$\because \log e = 1$}

Hence,

$$\lim_{x\to0} \frac{e^{3+x} - \sin x - e^3}{x} = e^3 - 1$$

34. Question

Evaluate the following limits:

$$\lim_{x\to0} \frac{e^x - x - 1}{2}$$

Answer

As we need to find $\lim_{x\to0} \frac{e^x-x-1}{2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to0} \frac{e^x-x-1}{2} = \frac{e^0-0-1}{2} = \frac{1-1}{2} = 0$ (not indeterminate)

As we got a finite value, so no need to do any modifications.

Hence,

$$\lim_{x\to0} \frac{e^x - x - 1}{2} = 0$$

35. Question

Evaluate the following limits:

$$\lim_{x\to0} \frac{e^{3x} - e^{2x}}{x}$$

Answer

As we need to find $\lim_{x\to0} \frac{e^{3x}-e^{2x}}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to0} \frac{e^{3x}-e^{2x}}{x} = \frac{e^0-e^0}{0} = \frac{1-1}{0} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

As Z = $\lim_{x\to 0}\frac{e^{3x}-e^{2x}}{x}$

⇒ Z = $\lim_{x\to 0}\frac{e^{3x}-1-e^{2x}+1}{x}$

{Adding and subtracting 1 in numerator}

⇒ Z = $\lim_{x\to 0}\frac{e^{3x}-1}{x} - \lim_{x\to 0}\frac{e^{2x}-1}{x}$

{using algebra of limits}

To get the form as present in the formula we multiply and divide 3 and 2 into both terms respectively:

⇒ Z = $3\lim_{x\to 0}\frac{e^{3x}-1}{3x} - 2\lim_{x\to 0}\frac{e^{2x}-1}{2x}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

∴ Z = 3log e – 2log e= 3-2 = 1

{using log e = 1}

Hence,

$$\lim_{x\to 0}\frac{e^{3x}-e^{2x}}{x} = 1$$

36. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{e^{\tan x}-1}{\tan x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{e^{\tan x}-1}{\tan x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to 0}\frac{e^{\tan x}-1}{\tan x} = \frac{e^0-1}{\tan 0} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

As, x→ 0

∴ tan x → 0

Let, y = tan x

∴ if x→ 0 ⇒ y→0

Hence, Z can be rewritten as-

$$\lim_{y\to 0}\frac{(e^y-1)}{y}$$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

$\therefore$ Z = log e = 1

{$\because$ log e = 1}

Hence,

$$\lim_{x\to 0}\frac{e^{\tan x}-1}{\tan x}=1$$

37. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{e^{bx}-e^{\sin x}}{x-\sin x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{e^{bx}-e^{\sin x}}{bx-\sin x}$

We can directly find the limiting value of a function by putting the value of variable at which the limiting value is asked, if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc)

Let Z = $\lim_{x\to 0}\frac{e^{bx}-e^{\sin x}}{bx-\sin x}=\frac{e^{0}-e^{\sin 0}}{0-\sin 0}=\frac{1-1}{0}$ (indeterminate)

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^{x}-1)}{x}=\log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

This question is a direct application of limits formula of exponential limits.

As Z = $\lim_{x\to 0}\frac{e^{bx}-e^{\sin x}}{bx-\sin x}$

$\Rightarrow$ Z = $\lim_{x\to 0}\frac{e^{\sin x}\left(\frac{e^{bx}}{e^{\sin x}}-1\right)}{bx-\sin x}$

$\Rightarrow$ Z = $\lim_{x\to 0}\frac{e^{\sin x}(e^{bx-\sin x}-1)}{bx-\sin x}$

{using properties of exponents}

$\Rightarrow$ Z = $\lim_{x\to 0}e^{\sin x}\times\lim_{x\to 0}\frac{(e^{bx-\sin x}-1)}{bx-\sin x}$

{using algebra of limits}

$\Rightarrow$ Z = $e^{\sin 0}\times\lim_{x\to 0}\frac{(e^{bx-\sin x}-1)}{bx-\sin x}=e^{0}\times\lim_{x\to 0}\frac{(e^{bx-\sin x}-1)}{bx-\sin x}$

$\therefore$ Z = $\lim_{x\to 0}\frac{(e^{bx-\sin x}-1)}{bx-\sin x}$

As, x→ 0

$\therefore$ bx-sin x → 0

Let, y = bx-sin x

$\therefore$ if x→0 ⇒ y→0

Hence, Z can be rewritten as-

Z = $\lim_{y\to 0}\frac{(e^{y}-1)}{y}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$

∴ Z = log e =1

{∵ log e = 1}

Hence,

$$\lim_{x\to 0}\frac{e^{bx}-e^{\sin x}}{bx-\sin x} = 1$$

38. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$$

Answer

As we need to find $\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to 0}\frac{e^{\tan x}-1}{x} = \frac{e^0-1}{0} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

∵ Z = $\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$

To get the desired form, we proceed as follows-

Dividing numerator and denominator by tan x-

⇒ Z = $\lim_{x\to 0}\frac{\frac{e^{\tan x}-1}{\tan x}}{\frac{x}{\tan x}}$

Using algebra of limits-

Z = $\lim_{x\to 0}\frac{e^{\tan x}-1}{\tan x}\times\lim_{x\to 0}\frac{\tan x}{x}$

Use the formula - $\lim_{x\to 0}\frac{\tan x}{x} = 1$ (sandwich theorem)

∴ Z = $\lim_{x\to 0}\frac{e^{\tan x}-1}{\tan x}\times 1 = \lim_{x\to 0}\frac{e^{\tan x}-1}{\tan x}$

As, x→ 0

∴ tan x → 0

Let, y = tan x

∴ if x→ 0 ⇒ y→0

Hence, Z can be rewritten as-

$\lim\limits_{y\to 0}\frac{(e^y-1)}{y}$

Use the formula: $\lim\limits_{x\to 0}\frac{(a^x-1)}{x}=\log a$

∴ Z = log e = 1

{∵ log e = 1}

Hence,

$\lim\limits_{x\to 0}\frac{e^{\tan x}-1}{x}=1$

39. Question

Evaluate the following limits:

$$\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}$$

Answer

As we need to find $\lim\limits_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim\limits_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}=\frac{e^0-e^{\sin o}}{0-\sin 0}=\frac{1-1}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula$\lim\limits_{x\to 0}\frac{(a^x-1)}{x}=\log a$ and $\lim\limits_{x\to 0}\frac{\log(1+x)}{x}=1$

This question is a direct application of limits formula of exponential limits.

As Z = $\lim\limits_{x\to 0}\frac{e^x-e^{\sin x}}{bx-\sin x}$

⇒ Z = $\lim\limits_{x\to 0}\frac{e^{\sin x}(\frac{e^x}{e^{\sin x}}-1)}{x-\sin x}$

⇒ Z = $\lim\limits_{x\to 0}\frac{e^{\sin x}(e^{x-\sin x}-1)}{x-\sin x}$

{using properties of exponents}

⇒ Z = $\lim\limits_{x\to 0}e^{\sin x}\times\lim\limits_{x\to 0}\frac{(e^{x-\sin x}-1)}{x-\sin x}$

{using algebra of limits}

⇒ Z = $e^{\sin 0}\times\lim\limits_{x\to 0}\frac{(e^{x-\sin x}-1)}{x-\sin x}=e^0\times\lim\limits_{x\to 0}\frac{(e^{x-\sin x}-1)}{x-\sin x}$

∴ Z = $\lim\limits_{x\to 0}\frac{(e^{x-\sin x}-1)}{x-\sin x}$

As, x→ 0

∴ x-sin x → 0

Let, y = x-sin x

∴ if x→0 ⇒ y→0

Hence, Z can be rewritten as-

$Z = \lim_{y\to0}\frac{(e^y-1)}{y}$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$

∴ Z = log e =1

{∵ log e = 1}

Hence,

$$\lim_{x\to0}\frac{e^x - e^{\sin x}}{x - \sin x} = 1$$

40. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{3^{2+x} - 9}{x}$$

Answer

As we need to find $\lim_{x\to0}\frac{3^{2+x}-9}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let $Z = \lim_{x\to0}\frac{3^{x+2}-3^2}{x} = \frac{3^2-3^2}{0} == \frac{0}{0}$ (indeterminate)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula$\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

As $Z = \lim_{x\to0}\frac{3^{x+2}-3^2}{x}$

⇒ $Z = \lim_{x\to0}\frac{3^2(3^x-1)}{x}$

⇒ $Z = 9\lim_{x\to0}\frac{(3^x-1)}{x}$

{using algebra of limits}

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$

∴ Z = 9 log 3

Hence,

$$\lim_{x\to0}\frac{3^{x+2} - 9}{x} = 9\log_e 3$$

41. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{a^x - a^{-x}}{x}$$

Answer

As we need to find $\lim_{x\to0}\frac{a^x-a^{-x}}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to0}\frac{a^x-a^{-x}}{x}=\lim_{x\to0}\frac{a^0-a^{-0}}{0}=\frac{1-1}{0}=\frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x}=1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get the desired forms, we need to include mx and nx as follows:

As Z = $\lim_{x\to0}\frac{a^x-a^{-x}}{x}$

⇒ Z = $\lim_{x\to0}\frac{a^{-x}\left(\frac{a^x}{a^{-x}}-1\right)}{x}=\lim_{x\to0}\frac{a^{-x}(a^{2x}-1)}{x}$

{using law of exponents}

⇒ Z = $\lim_{x\to0}a^{-x}\times\lim_{x\to0}\frac{(a^{2x}-1)}{x}$

{using algebra of limits}

⇒ Z = $a^{-0}\times\lim_{x\to0}\frac{(a^{2x}-1)}{x}$

⇒ Z = $\lim_{x\to0}\frac{(a^{2x}-1)}{x}$

To get the form as present in the formula we multiply and divide by 2

∴ Z = $\lim_{x\to0}\frac{(a^{2x}-1)}{2x}\times2$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x}=\log a$

∴ Z = 2 log a

Hence,

$$\lim_{x\to0}\frac{a^x-a^{-x}}{x}=2\log_e a$$

42. Question

Evaluate the following limits:

$$\lim_{x\to0}\frac{x(e^x-1)}{1-\cos x}$$

Answer

As we need to find $\lim_{x\to0}\frac{x(e^x-1)}{1-\cos x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc)

Let Z = $\lim_{x\to0}\frac{x(e^x-1)}{1-\cos x} = \lim_{x\to0}\frac{0(e^0-1)}{1-\cos 0} = \frac{0}{1-1} = \frac{0}{0}$ (indeterminate form)

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula$\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x\to0}\frac{\sin x}{x} = 1$

As Z = $\lim_{x\to0}\frac{x(e^x-1)}{1-\cos x}$

As, 1-cos x = $2\sin^2(x/2)$

∴ Z = $\lim_{x\to0}\frac{x(e^x-1)}{2\sin^2\left(\frac{x}{2}\right)}$

⇒ Z = $\frac{1}{2}\lim_{x\to0}\frac{x(e^x-1)}{\sin^2\left(\frac{x}{2}\right)}$

To get the desired form to apply the formula we need to divide numerator and denominator by x^2.

⇒ Z = $\frac{1}{2}\lim_{x\to0}\frac{\frac{x(e^x-1)}{x^2}}{\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2\times4}} = \frac{4}{2}\lim_{x\to0}\frac{\frac{(e^x-1)}{x}}{\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2}$

Using algebra of limits, we have-

Z = $2\frac{\lim_{x\to0}\frac{(e^x-1)}{x}}{\lim_{x\to0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2}$

Use the formula: $\lim_{x\to0}\frac{(a^x-1)}{x} = \log a$ and $\lim_{x\to0}\frac{\sin x}{x} = 1$

∴ Z = $2\,\frac{\log e}{1^2}$

⇒ Z = 2 log e = 2

Hence,

$$\lim_{x\to0}\frac{x(e^x-1)}{1-\cos x} = 2$$

43. Question

Evaluate the following limits:

$$\lim_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)}$$

Answer

As we need to find $\lim_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

Let Z = $\lim_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)} = \frac{2^{-\cos\frac{\pi}{2}}-1}{\frac{\pi}{2}\left(\frac{\pi}{2}-\frac{\pi}{2}\right)} = \frac{0}{0}$ (indeterminate form)

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x\to 0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

As Z = $\lim_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)}$

⇒ Z = $\lim_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{\left(x-\frac{\pi}{2}\right)}\times\lim_{x\to\frac{\pi}{2}}\frac{1}{x}$ {using algebra of limits}

⇒ Z = $\lim_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{\left(x-\frac{\pi}{2}\right)}\times\frac{1}{\frac{\pi}{2}}=\frac{2}{\pi}\lim_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{\left(x-\frac{\pi}{2}\right)}$

⇒ Z = $\frac{2}{\pi}\lim_{x\to\frac{\pi}{2}}\frac{2^{\sin\left(x-\frac{\pi}{2}\right)}-1}{\left(x-\frac{\pi}{2}\right)}$ {∵ sin(x-π/2) = -cos x}

As x→π/2

∴ x-π/2→0

Let x-π/2 = y and y→0

Z can be rewritten as-

Z = $\frac{2}{\pi}\lim_{y\to 0}\frac{2^{\sin(y)}-1}{y}$

Dividing numerator and denominator by sin y to get the form present in the formula

Z = $\frac{2}{\pi}\lim_{y\to 0}\frac{\frac{2^{\sin(y)}-1}{\sin y}}{\frac{y}{\sin y}}$

Using algebra of limits:

Z = $\frac{2}{\pi}\lim_{y\to 0}\frac{2^{\sin y}-1}{\sin y}\times\lim_{y\to 0}\frac{\sin y}{y}$

Use the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x}=\log a$ and $\lim_{x\to 0}\frac{\sin x}{x}=1$

∴ Z = $\frac{2}{\pi}\log_e 2$

Hence,

$$\lim_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)}=\frac{2}{\pi}\log_e 2$$

Exercise 29.11

1. Question

Evaluate the following limits:

$$\lim_{x\to\pi}\left(1-\frac{x}{\pi}\right)^{\pi}$$

Answer

As we need to find $\lim_{x\to\pi}\left(1-\frac{x}{\pi}\right)^{\pi}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞,0^{∞} .. etc.)

Let Z = $\lim_{x\to\pi}\left(1-\frac{x}{\pi}\right)^{\pi} = \left(1-\frac{\pi}{\pi}\right)^{\pi} = (1-1)^{\pi} = 0^{\pi} = 0$

As it is not taking any indeterminate form.

∴ Z = 0

Hence,

$$\lim_{x\to\pi}\left(1-\frac{x}{\pi}\right)^{\pi} = 0$$

2. Question

Evaluate the following limits:

$$\lim_{x\to 0^{+}}\left\{1+\tan^{\sqrt{x}}\right\}^{1/2x}$$

Answer

As we need to find $\lim_{x\to 0^{+}}\{1+\tan^2\sqrt{x}\}^{1/2x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, 1^{∞} .. etc.)

Let Z = $\lim_{x\to 0^{+}}\{1+\tan^2\sqrt{x}\}^{1/2x} = \{1+\tan^2\sqrt{0}\}^{1/0} = (1)^{\infty}$ (indeterminate)

As it is taking indeterminate form.

∴ we need to take steps to remove this form so that we can get a finite value.

As, Z = $\lim_{x\to 0^{+}}\{1+\tan^2\sqrt{x}\}^{\frac{1}{2x}}$

⇒ Z = $\lim_{x\to 0^{+}}\{1+\tan^2\sqrt{x}\}^{\frac{1}{2x}}$

Taking log both sides-

⇒ log Z = $\lim_{x\to 0^{+}}\{1+\tan^2\sqrt{x}\}^{\frac{1}{2x}}$

⇒ log Z = $\lim_{x\to 0^{+}}\frac{\log(1+\tan^2\sqrt{x})}{2x}$

{∵ log a^m = m log a}

Now it gives us a form that can be reduced to $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

Dividing numerator and denominator by $\tan^2\sqrt{x}$ –

log Z = $\lim_{x\to 0^{+}}\frac{\frac{\log(1+\tan^2\sqrt{x})}{\tan^2\sqrt{x}}}{\frac{2x}{\tan^2\sqrt{x}}}$

using algebra of limits –

$$\log Z = \frac{\lim_{x\to 0^{+}}\frac{\log(1+\tan^2\sqrt{x})}{\tan^2\sqrt{x}}}{\lim_{x\to 0^{+}}\frac{2x}{\tan^2\sqrt{x}}} = \frac{A}{B}$$

$$A = \lim_{x\to 0^{+}}\frac{\log(1+\tan^2\sqrt{x})}{\tan^2\sqrt{x}}$$

Let, $\tan^2\sqrt{x} = y$

As $x \to 0^+ \Rightarrow y \to 0^+$

$\therefore A = \lim_{y\to 0} \frac{\log(1+y)}{y}$

Use the formula - $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

$\therefore A = 1$

Now, $B = \lim_{x\to 0^+} \frac{2x}{\tan^2\sqrt{x}}$

$\Rightarrow B = 2\lim_{x\to 0^+}\left(\frac{\sqrt{x}}{\tan\sqrt{x}}\right)^2$

Use the formula - $\lim_{x\to 0} \frac{\tan x}{x} = 1$

$\therefore B = 2$

Hence,

$\log Z = \frac{A}{B} = \frac{1}{2}$

$\Rightarrow \log_e Z = 1/2$

$\therefore Z = e^{1/2}$

Hence,

$$\lim_{x\to 0^+}\{1+\tan^2\sqrt{x}\}^{1/2x} = \sqrt{e}$$

3. Question

Evaluate the following limits:

$$\lim_{x\to 0} (\cos x)^{1/\sin x}$$

Answer

As we need to find $\lim_{x\to 0}(\cos x)^{1/\sin x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞,1^∞ .. etc.)

Let $Z = \lim_{x\to 0}(\cos x)^{1/\sin x} = \{\cos 0\}^{\frac{1}{\sin o}} = (1)^\infty$ (indeterminate)

As it is taking indeterminate form-

∴ we need to take steps to remove this form so that we can get a finite value.

As, $Z = \lim_{x\to 0}(\cos x)^{1/\sin x}$

$\Rightarrow Z = \lim_{x\to 0}(\cos x)^{1/\sin x}$

Taking log both sides-

$\Rightarrow \log Z = \lim_{x\to 0}\log(\cos x)^{1/\sin x}$

$\Rightarrow \log Z = \lim_{x\to 0}\left\{\frac{\log\cos x}{\sin x}\right\}$

{∵ $\log a^m = m \log a$}

Now it gives us a form that can be reduced to $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

$\log Z = \lim_{x \to 0} \left\{\frac{\log(1+\cos x - 1)}{\sin x}\right\}$ {adding and subtracting 1 to cos x to get the form}

Dividing numerator and denominator by cos x – 1 to match with form in formula

$\therefore \log Z = \lim_{x \to 0} \left\{\frac{\frac{\log(1+\cos x-1)}{\cos x - 1}}{\frac{\sin x}{\cos x - 1}}\right\}$

using algebra of limits –

$\log Z = \frac{\lim_{x \to 0} \frac{\log(1+\cos x-1)}{\cos x - 1}}{\lim_{x \to 0} \frac{\sin x}{\cos x - 1}} = \frac{A}{B}$

$\therefore A = \lim_{x \to 0} \frac{\log(1+\cos x-1)}{\cos x - 1}$

Let, cos x - 1 = y

As $x \to 0 \Rightarrow y \to 0$

$\therefore A = \lim_{y \to 0} \frac{\log(1+y)}{y}$

Use the formula - $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

$\therefore A = 1$

Now, $B = \lim_{x \to 0} \frac{\sin x}{\cos x - 1}$

$\because \cos x - 1 = -2\sin^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$

$\Rightarrow B = \lim_{x \to 0} \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{-2\sin^2\left(\frac{x}{2}\right)} = -\lim_{x \to 0} \cot\frac{x}{2}$

$\therefore B = -\cot 0 = \infty$

$\therefore B = \infty$

Hence,

$\log Z = \frac{A}{B} = \frac{1}{\infty} = 0$

$\Rightarrow \log_e Z = 0$

$\therefore Z = e^0 = 1$

Hence,

$\lim_{x \to 0} (\cos x)^{1/\sin x} = 1$

4. Question

Evaluate the following limits:

$$\lim_{x \to 0} (\cos x + \sin x)^{1/x}$$

Answer

As we need to find $\lim_{x \to 0} (\cos x + \sin x)^{1/x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞,1^∞ .. etc.)

Let Z = $\lim_{x\to 0}(\cos x + \sin x)^{\frac{1}{x}} = \{\cos 0 + \sin 0\}^{\frac{1}{0}} = (1)^{\infty}$ (indeterminate)

As it is taking indeterminate form-

∴ we need to take steps to remove this form so that we can get a finite value.

As, Z = $\lim_{x\to 0}(\cos x + \sin x)^{\frac{1}{x}}$

⇒ Z = $\lim_{x\to 0}(\cos x + \sin x)^{\frac{1}{x}}$

Taking log both sides-

⇒ log Z = $\lim_{x\to 0}\log(\cos x + \sin x)^{\frac{1}{x}}$

⇒ log Z = $\lim_{x\to 0}\left\{\frac{\log(\cos x+\sin x)}{x}\right\}$

{∵ log a^m = m log a}

Now it gives us a form that can be reduced to $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

log Z = $\lim_{x\to 0}\left\{\frac{\log(1+\cos x+\sin x-1)}{x}\right\}$

{adding and subtracting 1 to cos x to get the form}

Dividing numerator and denominator by cos x + sin x– 1 to match with form in formula

∴ log Z = $\lim_{x\to 0}\left\{\frac{\frac{\log(1+\cos x+\sin x-1)}{\cos x+\sin x-1}}{\frac{\sin x}{\cos x+\sin x-1}}\right\}$

using algebra of limits –

log Z = $\frac{\lim_{x\to 0}\frac{\log(1+\cos x+\sin x-1)}{\sin x+\cos x-1}}{\lim_{x\to 0}\frac{x}{\cos x+\sin x-1}} = \frac{A}{B}$

∴ A = $\lim_{x\to 0}\frac{\log(1+\cos x+\sin x-1)}{\sin x+\cos x-1}$

Let, cos x + sin x - 1 = y

As x→0 ⇒ y→0

∴ A = $\lim_{y\to 0}\frac{\log(1+y)}{y}$

Use the formula - $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

∴ A = 1

Now, B = $\lim_{x\to 0}\frac{x}{\cos x+\sin x-1}$

∵ cos x – 1 = -2sin^2(x/2) and sin x = 2sin(x/2)cos(x/2)

⇒ B = $\lim_{x\to 0}\frac{x}{-2\sin^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$

⇒ B = $\lim_{x\to 0}\frac{x}{2\sin\left(\frac{x}{2}\right)\{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\}}$

⇒ B = $\lim_{x\to 0}\frac{\frac{x}{2}}{\sin\left(\frac{x}{2}\right)} \times \lim_{x\to 0}\frac{1}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}$

Use the formula - $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\Rightarrow B = \lim_{x \to 0} \frac{1}{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)} = \frac{1}{\cos 0 - \sin 0}$

$\therefore B = 1$

Hence,

$\log Z = \frac{A}{B} = \frac{1}{1} = 1$

$\Rightarrow \log_e Z = 1$

$\therefore Z = e^1 = e$

Hence,

$\lim_{x \to 0} (\cos x + \sin x)^{1/x} = e$

5. Question

Evaluate the following limits:

$$\lim_{x \to 0} (\cos x + a \sin x)^{1/x}$$

Answer

As we need to find $\lim_{x \to 0} (\cos x + a \sin x)^{1/x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞,1^∞ .. etc.)

Let $Z = \lim_{x \to 0} (\cos x + a \sin x)^{\frac{1}{x}} = \{\cos 0 + a \sin 0\}^{\frac{1}{0}} = (1)^\infty$ (indeterminate)

As it is taking indeterminate form-

∴ we need to take steps to remove this form so that we can get a finite value.

As, $Z = \lim_{x \to 0} (\cos x + a \sin x)^{\frac{1}{x}}$

$\Rightarrow Z = \lim_{x \to 0} (\cos x + a \sin x)^{\frac{1}{x}}$

Taking log both sides-

$\Rightarrow \log Z = \lim_{x \to 0} \log(\cos x + a \sin x)^{\frac{1}{x}}$

$\Rightarrow \log Z = \lim_{x \to 0} \left\{\frac{\log(\cos x + a \sin x)}{x}\right\}$

$\{\because \log a^m = m \log a\}$

Now it gives us a form that can be reduced to $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

Adding and subtracting 1 to cos x to get the form-

$\log Z = \lim_{x \to 0} \left\{\frac{\log(1 + \cos x + a \sin x - 1)}{x}\right\}$

Dividing numerator and denominator by cos x + a sin x– 1 to match with form in formula

$\therefore \log Z = \lim_{x \to 0} \left\{\frac{\frac{\log(1 + \cos x + a \sin x - 1)}{\cos x + a \sin x - 1}}{\frac{x}{\cos x + a \sin x - 1}}\right\}$

using algebra of limits –

$\log Z = \frac{\lim_{x\to 0}\frac{\frac{\log(1+\cos x+a\sin x-1)}{a\sin x+\cos x-1}}{x}}{\lim_{x\to 0}\frac{x}{\cos x+a\sin x-1}} = \frac{A}{B}$

$\therefore A = \lim_{x\to 0}\frac{\log(1+\cos x+a\sin x-1)}{a\sin x+\cos x-1}$

Let, cos x + asin x - 1 = y

As $x\to 0 \Rightarrow y\to 0$

$\therefore A = \lim_{y\to 0}\frac{\log(1+y)}{y}$

Use the formula - $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

$\therefore A = 1$

Now, $B = \lim_{x\to 0}\frac{x}{\cos x+a\sin x-1}$

$\because \cos x - 1 = -2\sin^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$

$\Rightarrow B = \lim_{x\to 0}\frac{x}{-2\sin^2\left(\frac{x}{2}\right)+2a\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$

$\Rightarrow B = \lim_{x\to 0}\frac{x}{2\sin\left(\frac{x}{2}\right)\{a\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\}}$

$\Rightarrow B = \lim_{x\to 0}\frac{\frac{x}{2}}{\sin\left(\frac{x}{2}\right)} \times \lim_{x\to 0}\frac{1}{a\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}$

Use the formula - $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\Rightarrow B = \lim_{x\to 0}\frac{1}{a\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)} = \frac{1}{a\cos 0-\sin 0}$

$\therefore B = 1/a$

Hence,

$\log Z = \frac{A}{B} = \frac{1}{\frac{1}{a}} = a$

$\Rightarrow \log_e Z = a$

$\therefore Z = e^a = e^a$

Hence,

$\lim_{x\to 0}(\cos x + a\sin x)^{\frac{1}{x}} = e^a$

6. Question

Evaluate the following limits:

$$\lim_{x\to\infty}\left\{\frac{x^2+2x+3}{2x^2+x+5}\right\}^{\frac{3x-2}{3x+2}}$$

Answer

As we need to find $\lim_{x\to\infty}\left\{\frac{x^2+2x+3}{2x^2+x+5}\right\}^{\frac{3x-2}{3x+2}}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞,1^∞ .. etc.)

Let Z = $\lim_{x\to\infty}\left\{\frac{x^2+2x+3}{2x^2+x+5}\right\}^{\frac{3x-2}{3x+2}} = \left(\frac{\infty}{\infty}\right)^{\frac{\infty}{\infty}}$ (indeterminate)

As it is taking indeterminate form-

∴ we need to take steps to remove this form so that we can get a finite value.

$Z = \lim_{x\to\infty}\left\{\frac{x^2+2x+3}{2x^2+x+5}\right\}^{\frac{3x-2}{3x+2}}$

Take the log to bring the term in the product so that we can solve it more easily.

Taking log both sides-

$\log Z = \lim_{x\to\infty}\log\left\{\frac{x^2+2x+3}{2x^2+x+5}\right\}^{\frac{3x-2}{3x+2}}$

$\Rightarrow \log Z = \lim_{x\to\infty}\left(\frac{3x-2}{3x+2}\right)\log\left(\frac{x^2+2x+3}{2x^2+x+5}\right)$

{∵ $\log a^m = m \log a$}

$\Rightarrow \log Z = \lim_{x\to\infty}\left(\frac{3x-2}{3x+2}\right)\times\lim_{x\to\infty}\log\left(\frac{x^2+2x+3}{2x^2+x+5}\right)$

{using algebra of limits}

Still, if we put x = ∞ we get an indeterminate form,

Take the highest power of x common and try to bring x in the denominator of a term so that if we put x = ∞ term reduces to 0.

$\therefore \log Z = \lim_{x\to\infty}\left(\frac{x\left(3-\frac{2}{x}\right)}{x\left(3+\frac{2}{x}\right)}\right)\times\lim_{x\to\infty}\log\left(\frac{x^2\left(1+\frac{2x}{x^2}+\frac{3}{x^2}\right)}{x^2\left(2+\frac{x}{x^2}+\frac{5}{x^2}\right)}\right)$

$\Rightarrow \log Z = \lim_{x\to\infty}\frac{3-\frac{2}{x}}{3+\frac{2}{x}}\times\lim_{x\to\infty}\log\frac{1+\frac{2}{x}+\frac{3}{x^2}}{2+\frac{1}{x}+\frac{5}{x^2}}$

$\Rightarrow \log Z = \frac{3-\frac{2}{\infty}}{3+\frac{2}{\infty}}\times\log\frac{1+\frac{2}{\infty}+\frac{3}{\infty^2}}{2+\frac{1}{\infty}+\frac{5}{\infty^2}}$

$\Rightarrow \log Z = \frac{3}{3}\times\log\frac{1}{2} = \log\frac{1}{2}$

$\therefore \log_e Z = \log\frac{1}{2}$

$\Rightarrow Z = 1/2$

Hence,

$$\lim_{x\to\infty}\left\{\frac{x^2+2x+3}{2x^2+x+5}\right\}^{\frac{3x-2}{3x+2}} = \frac{1}{2}$$

7. Question

Evaluate the following limits:

$$\lim_{x\to1}\left\{\frac{x^3+2x^2+x+1}{x^2+2x+3}\right\}^{\frac{1-\cos(x-1)}{(x-1)^2}}$$

Answer

As we need to find $\lim_{x\to1}\left\{\frac{x^3+2x^2+x+1}{x^2+2x+3}\right\}^{\frac{1-\cos(x-1)}{(x-1)^2}}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞,1^∞ .. etc.)

Let Z =$\lim_{x\to1}\left\{\frac{x^3+2x^2+x+1}{x^2+2x+3}\right\}^{\frac{1-\cos(x-1)}{(x-1)^2}} = \left(\frac{5}{6}\right)^{\frac{0}{0}}$ (indeterminate)

As it is taking indeterminate form-

∴ we need to take steps to remove this form so that we can get a finite value.

Z = $\lim_{x\to1}\left\{\frac{x^3+2x^2+x+1}{x^2+2x+3}\right\}^{\frac{1-\cos(x-1)}{(x-1)^2}}$

Take the log to bring the power term in the product so that we can solve it more easily.

Taking log both sides-

log Z = $\lim_{x\to1}\left\{\frac{x^3+2x^2+x+1}{x^2+2x+3}\right\}^{\frac{1-\cos(x-1)}{(x-1)^2}}$

⇒ log Z = $\lim_{x\to1}\frac{1-\cos(x-1)}{(x-1)^2}\log\left\{\frac{x^3+2x^2+x+1}{x^2+2x+3}\right\}$

{∵ log a^m = m log a}

using algebra of limits-

⇒ log Z = $\lim_{x\to1}\left(\frac{1-\cos(x-1)}{(x-1)^2}\right)\times\lim_{x\to1}\log\left\{\frac{x^3+2x^2+x+1}{x^2+2x+3}\right\}$

⇒ log Z = $\lim_{x\to1}\left(\frac{1-\cos(x-1)}{(x-1)^2}\right)\times\log\left(\frac{1^3+2.1^2+1+1}{1^2+2\times1+3}\right)$

⇒ log Z = $\log\frac{5}{6}\lim_{x\to1}\left(\frac{1-\cos(x-1)}{(x-1)^2}\right)$

As, 1-cos x = $2\sin^2(x/2)$

∴ log Z = $\log\frac{5}{6}\lim_{x\to1}\left(\frac{2\sin^2\frac{x-1}{2}}{(x-1)^2}\right)$

Let (x-1)/2 = y

As x→1 ⇒ y→0

∴ Z can be rewritten as

Log Z = $\log\frac{5}{6}\lim_{y\to0}\left(\frac{2\sin^2 y}{4y^2}\right)$

⇒ log Z = $\frac{1}{2}\log\frac{5}{6}\lim_{y\to0}\left(\frac{\sin y}{y}\right)^2$

Use the formula - $\lim_{x\to0}\frac{\sin x}{x} = 1$

∴ log Z = $\frac{1}{2}\log\frac{5}{6}\times 1 = \log\left(\frac{5}{6}\right)^{\frac{1}{2}}$

⇒ log Z = $\log\sqrt{\frac{5}{6}}$

∴ Z = $\sqrt{\frac{5}{6}}$

Hence,

$$\lim_{x\to1}\left\{\frac{x^3+2x^2+x+1}{x^2+2x+3}\right\}^{\frac{1-\cos(x-1)}{(x-1)^2}}=\sqrt{\frac{5}{6}}$$

8. Question

Evaluate the following limits:

$$\lim_{x\to0}\left\{\frac{e^x+e^{-x}-2}{x^2}\right\}^{1/x^2}$$

Answer

Let $y=\lim_{x\to0}\left\{\frac{e^x+e^{-x}-2}{x^2}\right\}^{\frac{1}{x^2}}$

Putting the limit, we get,

$$y=\left(\frac{0}{0}\right)^{\infty}$$

This is an indeterminate form, so we need to solve this limit. Taking log on both sides we get,

$$\log_e y=\log_e\lim_{x\to0}\frac{e^x+e^{-x}-2^{\frac{1}{x^2}}}{x^2}$$

$$y=e^{\lim_{x\to0}\frac{\left\{\frac{e^x+e^{-x}-2}{x^2}-1\right\}}{x^2}}$$

Now, applying L-Hospital's rule, we get,

$$y=e^{\lim_{x\to0}\frac{x^2\{e^x-e^{-x}\}-\{((e^x+e^{-x}-2)/x^2)-1\}4x^3}{x^4}}$$

Applying L-hospital rule again we get,

$$y=e^{\lim_{x\to0}\frac{1}{2}\{(\lim_{x\to0}(x+1))/\lim_{x\to0}(6+6x+x^2)\}}$$

$$y=e^{\frac{1}{12}}$$

9. Question

Evaluate the following limits:

$$\lim_{x\to a}\left\{\frac{\sin x}{\sin a}\right\}^{\frac{1}{x-a}}$$

Answer

As we need to find $\lim_{x\to a}\left\{\frac{\sin x}{\sin a}\right\}^{\frac{1}{x-a}}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞,1^{∞} .. etc.)

Let Z $=\lim_{x\to a}\left\{\frac{\sin x}{\sin a}\right\}^{\frac{1}{x-a}}=\left(\frac{\sin a}{\sin a}\right)^{\infty}=1^{\infty}$ (indeterminate)

As it is taking indeterminate form-

∴ we need to take steps to remove this form so that we can get a finite value.

$Z = \lim_{x \to a} \left\{\frac{\sin x}{\sin a}\right\}^{\frac{1}{x-a}}$

Take the log to bring the power term in the product so that we can solve it more easily.

Taking log both sides-

$\log Z = \lim_{x \to a} \left(\frac{1}{x-a}\right) \log\left\{\frac{\sin x}{\sin a}\right\}$

{∵ $\log a^m = m \log a$}

Now it gives us a form that can be reduced to $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

⇒ $\log Z = \lim_{x \to a} \left(\frac{1}{x-a}\right) \log\left\{1 + \frac{\sin x - \sin a}{\sin a}\right\}$

Dividing numerator and denominator by $\frac{\sin x - \sin a}{\sin a}$ to get the desired form and using algebra of limits we have-

$\log Z = \lim_{x \to a} \frac{\log\left\{1 + \frac{\sin x - \sin a}{\sin a}\right\}}{\frac{\sin x - \sin a}{\sin a}} \times \lim_{x \to a} \frac{\sin x - \sin a}{\sin a(x-a)}$

if we assume $\frac{\sin x - \sin a}{\sin a} = y$ then as x→a ⇒ y→ 0

⇒ $\log Z = \lim_{y \to 0} \frac{\log\{1+y\}}{y} \times \lim_{x \to a} \frac{\sin x - \sin a}{\sin a(x-a)}$

Use the formula- $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

∴ $\log Z = 1 \times \lim_{x \to a} \frac{\sin x - \sin a}{\sin a(x-a)}$

⇒ $\log Z = \lim_{x \to a} \frac{\sin x - \sin a}{\sin a(x-a)} = \frac{1}{\sin a} \lim_{x \to a} \frac{\sin x - \sin a}{(x-a)}$

Now it gives us a form that can be reduced to $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Try to use it. We are basically proceeding with a hit and trial attempt.

⇒ $\log Z = \frac{1}{\sin a} \lim_{x \to a} \frac{\sin(x-a+a) - \sin a}{(x-a)}$

∵ sin (A+B) = sin A cos B + cos A sin B

⇒ $\log Z = \frac{1}{\sin a} \lim_{x \to a} \frac{\sin(x-a)\cos a + \cos(x-a)\sin a - \sin a}{(x-a)}$

⇒ $\log Z = \frac{1}{\sin a} \lim_{x \to a} \frac{\sin(x-a)\cos a}{(x-a)} + \frac{1}{\sin a} \lim_{x \to a} \frac{\cos(x-a)\sin a - \sin a}{x-a}$

⇒ $\log Z = \frac{\cos a}{\sin a} \lim_{x \to a} \frac{\sin(x-a)}{(x-a)} + \frac{\sin a}{\sin a} \lim_{x \to a} \frac{\cos(x-a) - 1}{x-a}$

⇒ $\log Z = \cot a \lim_{x \to a} \frac{\sin(x-a)}{(x-a)} - 1 \lim_{x \to a} \frac{2\sin^2\frac{x-a}{2}}{\left(\frac{x-a}{2}\right)^2} \times \frac{(x-a)}{4}$

Use the formula- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

⇒ log Z = cot a – 0

∴ log Z = cot a

∴ $Z = e^{\cot a}$

Hence,

$$\lim_{x\to a}\left\{\frac{\sin x}{\sin a}\right\}^{\frac{1}{x-a}} = e^{\cot a}$$

10. Question

Evaluate the following limits:

$$\lim_{x\to\infty}\left\{\frac{3x^2+1}{4x^2-1}\right\}^{\frac{x^3}{1+x}}$$

Answer

As we need to find $\lim_{x\to\infty}\left\{\frac{3x^2+1}{4x^2-1}\right\}^{\frac{x^3}{1+x}}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞,1^{∞} .. etc.)

Let Z $=\lim_{x\to\infty}\left\{\frac{3x^2+1}{4x^2-1}\right\}^{\frac{x^3}{1+x}} = \left(\frac{\infty}{\infty}\right)^{\frac{\infty}{\infty}}$ (indeterminate)

As it is taking indeterminate form-

∴ we need to take steps to remove this form so that we can get a finite value.

Z = $\lim_{x\to\infty}\left\{\frac{3x^2+1}{4x^2-1}\right\}^{\frac{x^3}{1+x}}$

Take the log to bring the term in the product so that we can solve it more easily.

Taking log both sides-

log Z = $\lim_{x\to\infty}\left\{\frac{3x^2+1}{4x^2-1}\right\}^{\frac{x^3}{1+x}}$

⇒ log Z = $\lim_{x\to\infty}\left(\frac{x^3}{1+x}\right)\log\left(\frac{3x^2+1}{4x^2-1}\right)$

{∵ log a^m = m log a}

⇒ log Z = $\lim_{x\to\infty}\left(\frac{x^3}{1+x}\right)\times\lim_{x\to\infty}\log\left(\frac{3x^2+1}{4x^2-1}\right)$

{using algebra of limits}

Still, if we put x = ∞ we get an indeterminate form,

Take highest power of x common and try to bring x in denominator of a term so that if we put x = ∞ term reduces to 0.

∴ log Z = $\lim_{x\to\infty}\left(\frac{x^3}{x\left(1+\frac{1}{x}\right)}\right)\times\lim_{x\to\infty}\log\left(\frac{x^2\left(3+\frac{1}{x^2}\right)}{x^2\left(4-\frac{1}{x^2}\right)}\right)$

⇒ log Z = $\lim_{x\to\infty}\frac{x^2}{1+\frac{1}{x}}\times\lim_{x\to\infty}\log\frac{3+\frac{1}{x^2}}{4-\frac{1}{x^2}}$

⇒ log Z = $\frac{\infty}{1+\frac{1}{\infty}}\times\log\frac{3+\frac{1}{\infty^2}}{4-\frac{1}{\infty^2}}$

⇒ log Z = $\log\frac{3}{4}\times\infty = -\infty$

{∵ log (3/4) is a negative value as 3/4<1}

$\therefore \text{Log}_e Z = -\infty$

$\Rightarrow Z = e^{-\infty} = 0$

Hence,

$$\lim_{x \to \infty} \left\{\frac{3x^2 + 1}{4x^2 - 1}\right\}^{\frac{x^3}{1+x}} = 0$$