XII - ISC Board

Mathematics - Question Paper Solutions

Date: 26.02.2018 **Max. Marks : 100**

SECTION - A (80 Marks)

Question 1

(i) The binary operation $*: R\times R\to R$ is defined as $a*b=2a+b$ Find $(2*3)*4$.

Ans. Given $a*b=2a+b$

$$(2*3)*4=(4+3)*4$$
$$=7*4$$
$$=14+4=18$$

(ii) If $A=\begin{pmatrix}5 & a\\ b & 0\end{pmatrix}$ and A is symmetric matrix, show that $a=b$

Ans. $A=\begin{pmatrix}5 & a\\ b & 0\end{pmatrix}$ given A is symmetric matrix

$$A=A^T$$

$$\begin{bmatrix}5 & a\\ b & 0\end{bmatrix}=\begin{bmatrix}5 & a\\ b & 0\end{bmatrix}^T=\begin{bmatrix}5 & b\\ a & 0\end{bmatrix}$$

$$\therefore a=b$$

(iii) Solve : $3\tan^{-1}x+\cot^{-1}x=\pi$

Ans. $3\tan^{-1}x+\cot^{-1}x=\pi$

$$\therefore\ 2\tan^{-1}x+\tan^{-1}x+\cot^{-1}x=\pi$$

$$\therefore\ 2\tan^{-1}x+\frac{\pi}{2}=\pi$$

$$2\tan^{-1}x=\pi-\frac{\pi}{2}$$

$$\tan^{-1}(x)=\frac{\pi}{2}-\frac{\pi}{4}$$

$$x=\tan\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$x=1$$

(iv) Without expanding at any stage, find the value of:

$$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

Ans. $$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

R_1 / R_1+R_3 , R_2 / R_2-R_3

$$=\begin{vmatrix} a+x & b+y & c+z \\ a+x & b+y & c+z \\ x & y & z \end{vmatrix}=0 \qquad (\because\ R_1=R_2)$$

(v) Find the value of constant 'k' so that the function $f(x)$ defined as:

$$f(x)=\begin{cases} \dfrac{x^2-2x-3}{x+1}, & x\neq -1 \\ k, & x=-1 \end{cases}$$

is continuous at $x=-1$.

Ans. Given $f(x)$ is continuous at $x=-1$

$$\therefore\ f(-1)=\lim_{x\to -1} f(x)$$

$$\therefore k=\lim_{x\to -1}\frac{x^2-2x-3}{x+1}$$

$$=\lim_{x\to -1}\frac{(x-3)(x+1)}{(x+1)} \qquad [\because x\to -1 \Rightarrow x+1\neq 0]$$

$$=-1-3=-4$$

$$\therefore K=-4$$

(vi) Find the approximate change in the volume 'V' of a cube of side x metres caused by decreasing the side by 1%.

Ans. Volume of a cube

$$V = x^3$$

$$\frac{dV}{dx} = 3x^2$$

$$\therefore \delta x = x \cdot \frac{1}{100} = \frac{-x}{100}$$

$\therefore$ Change in volume

$$\delta V = \left(\frac{dV}{dx}\right)\delta x \quad = (3x^2)\cdot\left(\frac{-x}{100}\right) \quad = -\left(\frac{3}{100}\right)x^3$$

$$= -\frac{3}{100}V = -V\frac{3}{100}$$

$\therefore$ change in volume decrease by 3%

(vii) Evaluate : $\int \frac{x^3+5x^2+4x+1}{x^2}dx.$

Ans. $I = \int \frac{x^3+5x^2+4x+1}{x^2}dx$

$$= \int\left[\frac{x^3}{x^2}+\frac{5x^2}{x^2}+\frac{4x}{x^2}+\frac{1}{x^2}\right]dx$$

$$= \int\left[x+5+\frac{4}{x}+\frac{1}{x^2}\right]dx$$

$$= \frac{x^2}{2}+5x+4\log|x|-\frac{1}{x}+c$$

(viii) Find the differential equation of the family of concentric circles $x^2 + y^2 = a^2$

Ans. Finally of concentric circles is $x^2 + y^2 = a^2$

$\therefore$ Differential w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore y\frac{dy}{dx} + x = 0$$

(ix) If A and B are events such that $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$ and $P(A\cap B)=\frac{1}{4}$, then find:

(a) $P(A/B)$

(b) $P(B/A)$

Ans. $P(A)=\frac{1}{2}$ $\quad P(B)=\frac{1}{3}$ $\quad P(A\cap B)=\frac{1}{4}$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{4}}{\frac{1}{3}}=\frac{3}{4}$$

$$P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{2}$$

(x) In a race, the probabilities of A and B winning the race are $\frac{1}{3}$ and $\frac{1}{6}$ respectively. Find the probability of neither of them winning the race.

Ans. Let A win the race be E_1

B win the race be E_2

$P(E_1)=\frac{1}{3}, \quad P(E_2)=\frac{1}{6}$

$$P(E_1'\cap E_2')=P(E_1')\cdot P(E_2')$$

$$=[1-P(E_1)][1-P(E_2)]$$

$$=\left(1-\frac{1}{3}\right)\left(1-\frac{1}{6}\right)$$

$$=\frac{2}{3}\times\frac{5}{6}=\frac{5}{9}$$

Question 2

If the function $f(x)=\sqrt{2x-3}$ is invertible then find its inverse. Hence prove that $(fof^{-1})(x)=x$

Ans. Let $y=\sqrt{2x-3}$

$\therefore y^2=2x-3$

$$x=\frac{y^2+3}{2}$$

$$\therefore f^{-1}(x)=\frac{x^2+3}{2}$$

Now,

$$L.H.S = fof^{-1}(x) = f[f^{-1}(x)]$$

$$= \sqrt{2f^{-1}(x) - 3}$$

$$= \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = x$$

$$\therefore fof'(x) = x$$

Question 3

If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$, prove that $a + b + c = abc$.

Ans. $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$

$$\tan^{-1} b + \tan^{-1} c = \pi - \tan^{-1}(a)$$

$$\tan^{-1}\left(\frac{b+c}{1-bc}\right) = \pi - \tan^{-1} a$$

$$\frac{b+c}{1-bc} = \tan(\pi - \tan^{-1} a)$$

$$\frac{b+c}{1-bc} = -\tan(\tan^{-1} a)$$

$$\frac{b+c}{1-bc} = -a$$

$$b + c = -a + abc$$

$$\therefore a + b + c = abc$$

Question 4

Use properties of determinants to solve for x:

$$\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0 \text{ and } x \neq 0.$$

Ans. $$\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0 \text{ and } x \neq 0.$$

$$C_1 / C_1 + (C_2 + C_3)$$

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix} = 0$$

$$(x+a+b+c)\begin{vmatrix}1 & b & c\\ 1 & x+b & a\\ 1 & b & x+c\end{vmatrix}=0$$

$R_1 | R_1 - R_3$

$$(x+a+b+c)\begin{vmatrix}0 & 0 & -x\\ 1 & x+b & a\\ 1 & b & x+c\end{vmatrix}=0$$

$\therefore (x+a+b+c)\left[0-0-x(b-x-b)\right]=0$

$(x+a+b+c)(x^2)=0$

$\therefore x^2=0$ or $x+a+b+c=0$

but $x \neq 0$

$\therefore x=-(a+b+c)$

Question 5

(a) Show that the function $f(x)=\begin{cases}x^2, & x\leq 1\\ \frac{1}{2}, & x>1\end{cases}$ is continuous at $x=1$ but not differentiable.

Ans. Continuity at $x=1$

$f(x=1)=x^2=(1)^2=1$

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{1}{x}=1$

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x^2=1$

$\therefore f(x=1)=\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=1$

$\therefore f(x)$ is continuous at $x=1$

Now differentiable at $x=1$

$$(R.H.D.\text{ at } x=1)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}$$

$$=\lim_{x\to 1}\frac{-(x-1)\frac{1}{x}}{(x-1)}$$

$$=-\frac{1}{1}=-1$$

$$(L.H.D.\text{ at } x=1)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1^-}\frac{x^2-1}{x-1}=2$$

$\therefore L.H.D. \neq R.H.D.$

$\therefore f(x)$ is not differentiable at $x=1$

OR

(b) Verify Rolle's theorem for the following function:

$f(x)=e^{-x}\sin x \; on \; [0,\pi]$

Ans. $f(x)=e^{-x}\cdot\sin x$

(i) $f(x)$ is continuous on $[0,\pi]$ because e^{-x} & $\sin x$ are continuous function on its domain.

(ii) e^{-x} and $\sin x$ is differentiable on $(0,\pi)$

(iii) $f(0)=e^{-0}\cdot\sin 0=0$

$f(\pi)=e^{-\pi}\cdot\sin\pi=0$

(iv) Let c be number such that $f'(c)=0$

$\therefore f'(x)=e^{-x}\cdot\cos x+\sin x\cdot e^{-x}(-1)$

$\therefore f'(c)=e^{-c}(\cos c-\sin c)$

$\therefore f'(c)=0$

$e^{-c}(\cos c-\sin c)=0$

$\therefore e^{-c}\neq 0 \therefore \cos C-\sin C=0$

$\tan C=1$

$$\therefore C=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\ldots\ldots$$

$$\therefore \frac{\pi}{4}\in[0,\pi]$$

$\therefore$ Rolle's theorem verify

Question 6

If $x=\tan\left(\frac{1}{a}\log y\right)$, prove that $\left(1+x^2\right)\frac{d^2y}{dx^2}+\left(2x-a\right)\frac{dy}{dx}=0$

Ans. $x=\tan\left(\frac{1}{a}\log y\right)$

$\therefore \quad \frac{1}{a}\log y = \tan^{-1} x$

differentiating both sides w.r.t. x

$y = e^{a\tan^{-1}x}$

$\frac{dy}{dx} = e^{a\tan^{-1}x}.\left(\frac{a}{1+x^2}\right)$

$\left(1+x^2\right)\frac{dy}{dx} = ay$

Again differentiating both sides w.r.t. x

$\left(1+x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx^2}\cdot 2x = a\frac{dy}{dx}$

$\left(1+x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(2x-a\right) = 0$

Question 7

Evaluate : $\int \tan^{-1}\sqrt{x}\,dx$

Ans. $I = \int \tan^{-1}\sqrt{x}\,dx$

Put $\sqrt{x} = t$

$\frac{1}{2\sqrt{x}}dx = dt$

$dx = 2\sqrt{x}dt \rightarrow dx = 2t\,dt$

$I = \int 2t\tan^{-1}t\,dt$

$I = 2\left[\frac{t^2}{2}\tan^{-1}t - \frac{1}{2}\int\frac{t^2}{1+t^2}dt\right]$

$I = 2\left[\frac{t^2}{2}\tan^{-1}t - \frac{1}{2}\int\left[\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}\right]dt\right]$

$I = \left[t^2\tan^{-1}t - t + \tan^{-1}t\right] + c$

$I = t^2\tan^{-1}t - t + \tan^{-1}t + c$

$I = (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + c$

Question 8

(a) Find the points on the curve $y = 4x^3 - 3x + 5$ at which the equation of the tangent is parallel to the x-axis.

Ans. $y = 4x^3 - 3x + 5 \qquad ...(i)$

$$\frac{dy}{dx} = 12x^2 - 3$$

Given that lines is parallel to x-a xis

$$\therefore \quad \frac{dy}{dx} = 0$$

$$12x^2 - 3 = 0$$

$$12x^2 = 3$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$

Put $x = \pm\frac{1}{2}$ in equation (i)

$x = \frac{1}{2}$ then , $y = 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right) + 5 = 4$

$\therefore$ Point $\left(\frac{1}{2}, 4\right)$

when $x = \frac{-1}{2}$ then $y = 4\left(\frac{-1}{2}\right)^3 - 3\left(\frac{-1}{2}\right) + 5 = 6$

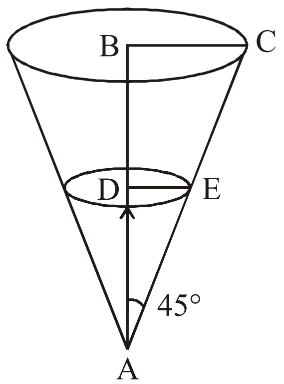
$\therefore$ Points $(x, y) = \left(\frac{1}{2}, 4\right)$ amd $\left(\frac{-1}{2}, 6\right)$

OR

(b) Water is dripping out from a conial funnel of semi-verticle angle $\frac{\pi}{4}$ at the uniform rate of 2 cm^2/sec in the surface, through a tiny hole at the vertex of the bottom. When the slant height of the water level is 4 cm, find the rate of decrease of the slant height of the water.

Ans. Let r be the radius, h be the height and V be the volume of the funnel at any time t.

$$V = \frac{1}{3}\pi r^2 h \qquad \text{...(i)}$$



Let I be the slant height of the funnel

Given : Semi-vertical angle = 45°

in the triangle ADE :

$$\sin 45° = \frac{DE}{AE} \Rightarrow \frac{1}{\sqrt{2}} = \frac{r}{l}$$

$$\cos 45° = \frac{AD}{AE} \Rightarrow \frac{1}{\sqrt{2}} = \frac{h}{l}$$

$r=\frac{1}{\sqrt{2}}$ and $h=\frac{1}{\sqrt{2}}$ ...(ii)

therefore the equation (i) can be rewritten as :

$$V=\frac{1}{3}\pi\times\left(\frac{I}{\sqrt{2}}\right)^2\times\frac{I}{\sqrt{2}}=\frac{\pi}{3\times2\times\sqrt{2}}\times I^3$$

$$V=\frac{\pi}{6\sqrt{2}}I^3 \quad \text{...(iii)}$$

Differentiate w.r.t. t :

$$\frac{dV}{dt}=\frac{\pi}{6\sqrt{2}}\times3l^2\times\frac{dl}{dt}$$

$$\frac{dV}{dt}=\frac{\pi}{2\sqrt{2}}\times l^2\times\frac{dl}{dt}$$

$$\frac{dl}{dt}=\frac{2\sqrt{2}}{\pi l^2}\times\frac{dV}{dt} \quad \text{...(iv)}$$

Since it is given that rate of change (decrease) of volume of water w.r.t. t is

$$\frac{dV}{dt}=-2cm^3/\sec$$

therefore

$$\frac{dl}{dt}=\frac{2\sqrt{2}}{\lambda l^2}\times(-2)=-\frac{4\sqrt{2}}{\lambda l^2}$$

$$\left.\frac{dl}{dt}\right|_{\text{at } l=4}=-\frac{4\sqrt{2}}{\pi\times(4)^2}=-\frac{\sqrt{2}}{4\pi}cm/\sec$$

Question 9

(a) Solve : $\sin x\frac{dy}{dx}-y=\sin x\cdot\tan\frac{x}{2}$

Ans. $\frac{dy}{dx}-y\cdot\operatorname{cosec}x=\tan\frac{x}{2}$

$$\frac{dy}{dx}-y\cdot\operatorname{cosec}x=\tan\frac{x}{2} \quad \text{...(i)}$$

Compare $\frac{dy}{dx}+Py=Q$

$P=-\operatorname{cosec}x, Q=\tan\frac{x}{2}$

$\text{I.F.}=e^{\int P\,dx}$

$\text{I.F.}=e^{\int -\operatorname{cosec}x\,dx}$

$\text{I.F.} = e^{-\log_e(\operatorname{cosec} x - \cot x)}$

$\text{I.F.} = e^{\log_e(\operatorname{cosec} x - \cot x)^{-1}}$

$\text{I.F.} = (\operatorname{cosec} x - \cot x)^{-1} = (\operatorname{cosec} x + \cot x)$

$\therefore$ solution of the linear differential equation

$y.IF. = \int Q.IF.dx$

$$y \cdot (\operatorname{cosec} x + \cot x) = \int \tan\frac{x}{2} \cdot \left(\frac{1+\cos x}{\sin x}\right) dx$$

$$y \cdot (\operatorname{cosec} x - \cot x) = \int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \cdot \frac{2 \cdot \cos^2\frac{x}{2}}{2\sin\frac{x}{2} \cdot \cos\frac{x}{2}} dx = \int 1\, dx$$

$y \cdot (\operatorname{cosec} x + \cot x) = x + c$

OR

(b) The population of a town grows at the rate of 10% per year. Using differential equation, find how long will it take for the population to grow 4 times.

Ans. Here $\frac{dx}{dt} \propto x$ [Since increase in population speeds up with increase in population] and let x be the population at anytime t.

$\therefore \frac{dx}{dt} = rx$ (where r is proportionality constant)

$\therefore \frac{dx}{x} = r \cdot dt$

integrating both sides

$\ln x = rt + c,$ (where c is the integration constant)

$\therefore x = e^{rt+c}$

$x = K e^{rt}$, where $K = e^c$

Here r is the rate of increase and K is the initial population let x_0 then $t = 0$

$x_0 = ke^o \Rightarrow k = x_0$

Given to find the time t taken to attain 4 times population, so $x = 4x_0$

So, $x = K e^{rt}$

$\Rightarrow 4x_0 = x_0\, e^{0.10t}$

$2 = e^{0.05t}$

Taking log on both sides

$\ln 2 = \ln e^{0.1t}$

$0.69314 = 0.1t \qquad t = 6.9314$

Question 10

(a) Using matrices, solve the following system of equations :

$2x-3y+5z=11$

$3x+2y-4z=-5$

$x+y-2z=-3$

Ans.

(a) Using this let us solve the system of given equation

$2x-3y+5x=11$

$3x+2y-4z=-5$

$x+y-2z=-3$

This can be written in the form AX = B

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

where $A=\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} B=\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

we know $A^{-1}=\frac{1}{|A|}(adj\ A)$

$|A|=2(2\times-2-1\times-4)-(-3)(3\times-2-1\times-4)+5(3\times1-2\times1)$

$=2(-4+4)+3(-6+4)+5(3-2)$

$=0-6+5=-1\neq0$

Hence it is a non - singular matrix

Therefore A^{-1} exists

Let us find the (adj A) by finding the minors and cofactors

$$M_{11}=\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix}=-4+4=0$$

$$M_{12}=\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix}=-6+4=-2$$

$$M_{13}=\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}=3-2=1$$

$$M_{21}=\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix}=6-5=1$$

$$M_{22}=\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix}=-4-5=-9$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 2+3 = 5$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 12-10 = 2$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -8-15 = -23$$

$$M_{33} = \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 4+9 = 13$$

$A_{11} = 0 \ A_{12} = 2 \ \ A_{13} = 1$
$A_{21} = -1 A_{22} = -9 \ A_{23} = -5$
$A_{31} = 2 \ \ A_{32} = 23 \ \ A_{33} = 13$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

We know AX = B, then $X = A^{-1}$ B

Therefore $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Matrix multiplication can be done by multiplying the rows of matrix A with the column of matrix B.

Therefore, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -5 & +6 \\ -22 & -45 & +69 \\ -11 & -25 & +39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Hence x=1 , y=2 and z = 3

OR

(b) Using elementary tranformation, find the inverse of the matrix :

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Ans. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$|A| = 0[2-3]-1[1-9]+2[1-6]$

$= 8 + 2(-5) = 8 - 10 = -2 \neq 0$

$\Rightarrow A^{-1}$ exists.

$\therefore A = IA$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 / R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_3 / R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$R_2 / R_2 - R_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 3 & -1 \\ 5 & -3 & 1 \end{bmatrix} A$$

$R_1 / R_1 - 2R_2$ and $R_3 \times \frac{1}{2}$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -5 & 2 \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

$R_1 / R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Question 11

A speaks truth in 60% of the cases, while B is 40% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact ?

Ans. A speaks truth $P\ (A=)\ \frac{60}{100}, P(A') = \frac{40}{100}$

B speaks truth $P(B) = \frac{40}{100}, P(B') = \frac{60}{100}$

they contradict each other $= P(A)\cdot(B') + P(A')\cdot P(B)$

$$= \frac{60}{100} \times \frac{60}{100} + \frac{40}{100} \times \frac{40}{100}$$

$$= \frac{3600 + 1600}{10000}$$

$$= \frac{5200}{10000}$$

$$= \frac{52}{100}$$

% of cases they likely to contradict each other $= \frac{52}{100} \times 100 = 52\%$

Question 12

A cone is inscribed in a sphere of radius 12 cm. If the volume of the cone is maximum, find its height.

Ans. Let VAB be a cone of greatest volume inscribed in a sphere of radius 12. It is obvious that for maximum volume the axis of the cone must be along a diameter of the sphere. Let VC be the axis of the cone and O be the centre of the sphere such that $OC = x$.

Then,

$VC = VO + OC = R + x = (12 + x) =$ height of cone

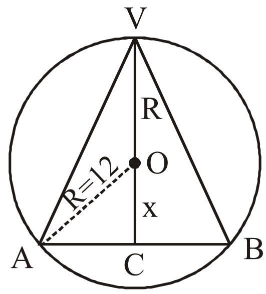
Applying Pythagoras theorem,

$OA^2 = AC^2 + OC^2$

$AC^2 = 12^2 - x^2$

$AC^2 = 144 - x^2$

Let V be the volume of the cone, then



$$V = \frac{1}{3}\pi(AC)^2(VC)$$

$$= \frac{1}{3}\pi(144 - x^2)(12 + x)$$

$$= \frac{1}{3}\pi\left[1728 + 144x - 12x^2 - x^3\right] \qquad \text{...(i)}$$

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[144 - 24x - 3x^2\right]$$

$$\frac{d^2V}{dx^2} = \frac{1}{3}\pi[-24 - 6x] = \frac{1}{3}\pi(-6)^2[4 + x] = -2\pi(4 + x)$$

Now, $\frac{dV}{dx} = 0$ gives $\frac{1}{3}\pi\left[144 - 24x - 3x^2\right] = 0$

i.e., $144 - 24x - 3x^2 = 0$

i.e., $x^2 + 8x - 48 = 0$

i.e., $(x + 12)(x - 4) = 0$

i.e., $x = -12$ or $x = 4$

$$\left[\frac{d^2V}{dx^2}\right]_{x=4} = -2\pi(4 + 4) = -16\pi < 0$$

Thus, V is maximum when $x = 4$

Putting $x = 4$ in (1), we obtain

$\therefore$ Height of the cone $= x + R = 4 + 12 = 16\,cm$

Question 13

(a) Evaluate : $\int \frac{x-1}{\sqrt{x^2 - x}}dx$

Ans. Let $I = \int\sqrt{\frac{x-1}{\sqrt{x^2-x}}}dx$

$$\therefore x - 1 = A\frac{d}{dx}(x^2 - x) + B$$

$$x - 1 = A(2x - 1) + B$$

$$1 = 2A \quad \Rightarrow \boxed{A = \frac{1}{2}}$$

$$-1 = -A + B \Rightarrow -1 = \frac{-1}{2} + B \Rightarrow B = \frac{-1}{2}$$

$$I=\int \frac{\frac{1}{2}(2x-1)dx}{\sqrt{x^2-x}}dx-\int \frac{1}{2}\frac{dx}{\sqrt{x^2-x}}dx$$

$$=\int \frac{\frac{1}{2}(2x-1)dx}{\sqrt{x^2-x}}-\frac{1}{2}\int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$=\frac{1}{2}\times 2\sqrt{x^2-x}-\frac{1}{2}\times \log\left|\left(x-\frac{1}{2}\right)+\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=\sqrt{x^2-x}-\frac{1}{2}\log\left|x-\frac{1}{2}+\sqrt{x^2-x}\right|+C$$

OR

(b) Evaluate : $\int_0^{\pi/2} \frac{\cos^2 x}{1+\sin x\cos x}dx$

Ans. $I=\int_0^{\pi/2} \frac{\cos^2 x}{1+\sin x\cos x}\cdot dx$ ...(1)

Using $\int_0^a f(x)\,dx=\int_0^a f(a-x)dx$

$$I=\int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$=\int_0^{\pi/2} \frac{\sin^2 x}{1+\cos x\cdot\sin x}dx \quad ...(2)$$

Adding eq. (1) & (2)

$$2I=\int_0^{\pi/2} \frac{\cos^2 x+\sin^2 x}{1+\sin x\cos x}dx$$

$$=\int_0^{\pi/2} \frac{1}{1+\sin x\cos x}dx$$

$$=\int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x+\tan x}dx$$

$$2I=\int_0^{\pi/2}\frac{\sec^2 x\,dx}{1+\tan^2 x+\tan x}$$

Put $\tan x=t,\ \sec^2 x\,dx=dt$

when $\quad x=0,\ t=0$

when $\quad x=\frac{\pi}{2},\ t=\infty$

$$2I=\int_0^{\infty}\frac{dt}{t^2+2t.\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+1}$$

$$=\int_0^{\infty}\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{\frac{\sqrt{3}}{2}}\left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^{\infty}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2t+1}{\sqrt{3}}\right]_0^{\infty}$$

$$2I=\frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right]$$

$$I=\frac{1}{\sqrt{3}}\left[\frac{3\pi-\pi}{6}\right]$$

$$=\frac{1}{\sqrt{3}}\left[\frac{2\pi}{6}\right]=\frac{\pi}{3\sqrt{3}}$$

Question 14

From a lot of 6 items containing 2 defective items, a sample of 4 items are drawn at random. Let the random variable X denote the number of defective items in the sample.

If the sample is drawn without replacement, find :

(a) The probability distribution of X

(b) Mean of X

(c) Variance of X

Ans. In 6 items 2 defective and 4 non-defective

Let P is the probability of defective items

Let $x=$ number of defective items

$\therefore\ x=0,1,2$

$$\therefore P(x=0) = \frac{{}^{4}C_{4}}{{}^{6}C_{4}} = \frac{1}{15}$$

$$\therefore P(x=1) = \frac{{}^{2}C_{1} \times {}^{4}C_{3}}{{}^{6}C_{4}} = \frac{8}{15}$$

$$\therefore P(x=2) = \frac{{}^{2}C_{2} \times {}^{4}C_{2}}{{}^{6}C_{4}} = \frac{6}{15}$$

X	$P(x)$	$xP(x)$	$x^2P(x)$
0	$\frac{1}{15}$	0	0
1	$\frac{8}{15}$	$\frac{8}{15}$	$\frac{8}{15}$
2	$\frac{6}{15}$	$\frac{12}{15}$	$\frac{24}{15}$

(b) Mean $\left(\overline{X}\right) = \sum P_i X_i$

$$= \frac{20}{15} = \frac{4}{3}$$

(c) Variance $\left(\sigma^2\right) = \sum P_i X_i^2 - \left(\sum P_i X_i\right)^2$

$$= \frac{32}{15} - \left(\frac{4}{3}\right)^2 = \frac{16}{45} = 0.35$$

SECTION - B (20 Marks)

Question 15

(a) Find λ if the scalar projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.

Ans. Projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 4$

$$= \frac{\left(\lambda\hat{i} + \hat{j} + 4\hat{k}\right)\cdot\left(2\hat{i} + 6\hat{j} + 3\hat{k}\right)}{\sqrt{2^2 + 6^2 + 3^2}} = 4$$

$$\therefore \frac{2\lambda + 6(1) + 4(3)}{\sqrt{49}} = 4$$

$\therefore \dfrac{2\lambda+18}{7}=4$

$\therefore 2\lambda = 28-18$

$\therefore 2\lambda = 10$

$\lambda = 5$

(b) The Cartesian equation of line is : $2x-3=3y+1=5-6z$. Find the vector equation of a line passing through (7, –5, 0) and parallel to the given line.

Ans. Cartesian equation of a line is

$2x-3=3y+1=5-6z$

i.e., $2\left(x-\frac{3}{2}\right)=3\left(y+\frac{1}{3}\right)=-6\left(z-\frac{5}{6}\right)$

Dividing by –6 throughout

i.e., $\dfrac{x-\frac{3}{2}}{-3}=\dfrac{y+\frac{1}{3}}{-2}=\dfrac{2-\frac{5}{6}}{1}$

$\therefore$ D.r.s of the above line is –3, –2, 1

Now, equation of a line passing through point (7, –5, 0) and parallel to the above line whose d.r.s. is –3, –2, 1 is

$\vec{r}=\left(7\hat{i}-5\hat{j}\right)+\lambda\left(-3\hat{i}-2\hat{j}+\hat{k}\right)$

$\therefore \vec{r}=\left(7\hat{i}-5\hat{j}\right)+\lambda\left(3\hat{i}+2\hat{j}-\hat{k}\right)$

(c) Find the equation of the plane through the intersection of the planes $\vec{r}\cdot\left(\hat{i}+3\hat{j}-\hat{k}\right)=9$ and $\vec{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=3$ and passing through the origin.

Ans. Equation of I plane is $\bar{r}\cdot\left(\hat{i}+3\hat{j}-\hat{k}\right)=9$

i.e., $\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(\hat{i}+3\hat{j}-\hat{k}\right)=9$

$x+3y-z=9$

$x+3y-z-9=0$ ...(i)

Equation of II plane is $\bar{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=3$

i.e., $\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=3$

i.e., $2x-y+z=3$

i.e., $2x-y+z-3=0$ ...(ii)

Now, equation of a plane passing through intersection of given planes is

$\left(x+3y-z-9\right)+\lambda\left(2x-y+z-3\right)=0$

$\left(1+2\lambda\right)x+\left(3-\lambda\right)y+\left(-1+\lambda\right)z-9-3\lambda=0$

Since plane is passing through the origin (0, 0, 0)

$-9-3\lambda=0$

$-3\lambda=9$

$\lambda=-3$

Question 16

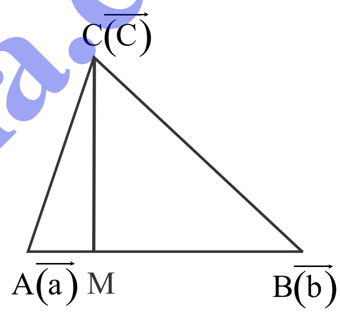
(a) If A, B, C are three non- collinear points with position vectors $\vec{a}, \vec{b}, \vec{c},$ respectively, then show that the length of the perpendicular from C on AB is $\dfrac{\left|(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})\right|}{|\vec{b}-\vec{a}|}$

Ans. Let ABC be a triangle and let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of its vertices A, B, C respectively. Let CM be the perpendicular from C on AB. Then,

$$\text{Area of } \Delta ABC=\frac{1}{2}(AB)\cdot CM=\frac{1}{2}|\overline{AB}|(CM)$$

$$\text{Also, Area of } \Delta ABC=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

$$\text{Area of } \Delta ABC=\frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$$



$$\therefore\ \frac{1}{2}\left|\overline{AB}\right|(CM)=\frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$$

$$\Rightarrow\ CM=\frac{\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|}{\left|\overline{AB}\right|}$$

$$\Rightarrow\ CM=\frac{\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|}{\left|\vec{b}-\vec{a}\right|}$$

OR

(b) Show that the four points A,B, C and D with position vectors $4\hat{i}+5\hat{j}+\hat{k},\ -\hat{j}-\hat{k},\ 3\hat{i}+9\hat{j}+4\hat{k}$ and $4\left(-\hat{i}+\hat{j}+\hat{k}\right)$ respectively, are coplanar.

Ans. $\bar{a}=4\hat{i}+5\hat{j}+\hat{k}$

$\bar{b}=-\hat{i}-\hat{j}$

$\bar{c}=3\hat{i}+9\hat{j}+4\hat{k}$

$\bar{d}=-4\hat{i}+4\hat{j}+4\hat{k}$

$\overline{AB}=\bar{b}-\bar{a}=-4\hat{i}-6\hat{j}-2\hat{k}$

$\overline{AC} = \bar{c} - \bar{a} = -\hat{i} + 4\hat{j} + 3\hat{k}$

$\overline{AD} = \bar{d} - \bar{a} = -8\hat{i} - \hat{j} + 3\hat{k}$

$\overline{AB}, \overline{AC}$ & $\overline{AD}$ are coplanar if $\left[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}\right] = 0$ i.e., $\overline{AB}.\left(\overline{AC} \times \overline{AD}\right) = 0$

$$= \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$= -4(12+3) + 6(-3+24) - 2(1+32)$

$= -4(15) + 6(21) - 2(66)$

$= -60 + 126 - 66$

$= 0$

$\therefore$ $\overline{AB}, \overline{AC}$ & $\overline{AD}$ are coplanar

$\therefore$ Points A, B, C and D are coplanar

Question 17

(a) Draw a rough sketch of the curve and find the area of the region bounded by curve $y^2 = 8x$ and the line $x = 2$.

Ans.

Given equation is $y^2 = 8x$

Comparing with $y^2 = 4ax$

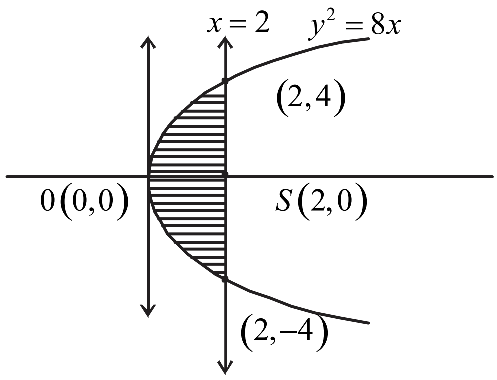
We get $4a = 8$

i.e. $a = 2$

Given $y^2 = 4(2)x$

$y^2 = 8x$

$\therefore y = \sqrt{8x}$



Also, $x = 2$ meets $y^2 = 8x$

$\therefore y^2 = 16$

$\therefore y = \pm 4$

$\therefore (2,4)$ and $(2,-4)$ are their point of intersection.

$\therefore$ Required are $A = 2\int_0^2 \sqrt{8x}\, dx$

$= 2\sqrt{8}\int_0^2 x^{1/2} dx$

$$= 4\sqrt{2} = \left[\frac{x^{3/2}}{3/2}\right]_0^2$$

$$= \frac{8\sqrt{2}}{3}\left[2^{3/2} - 0\right]$$

$$= \frac{8\sqrt{2}}{3} \times \sqrt{8}$$

$$= \frac{8\sqrt{2} \times 2\sqrt{2}}{3} = \frac{32}{3}\text{sq.units}$$

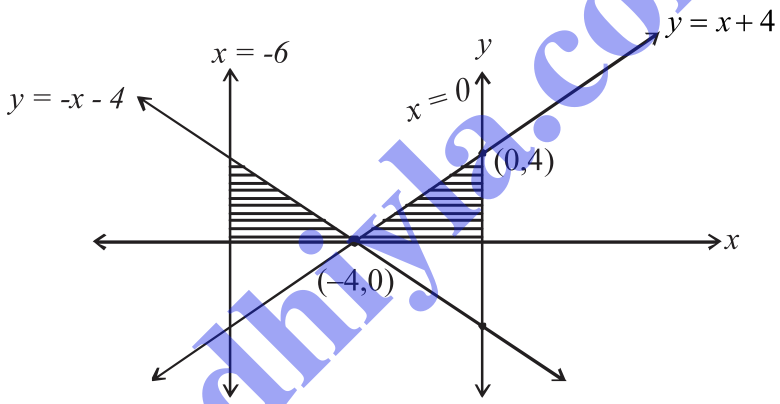
OR

(b) Sketch the graph of $y = |x+4|$. Using integration, find the area of the region bounded by the curve $y = |x + 4|$ and $x = -6$ and $x = 0$.

Ans.

$y = x + 4,$ if $x > 4$

& $y = -(x+4)$, if $x < 4$



For $y = x + 4$

when $x = 0, y = 4$

& when $y = 0, x = -4$

Points are

$\therefore (0,4)$ & $(-4,0)$

For $y = -x - 4$,

when $x = 0, y = -4$

when $y = 0, x = -4$

$\therefore$ Point are

$(0,-4)$ & $(-4,0)$

$\therefore$ Required area

$$= \int_{-6}^{-4} -(x+4)\,dx + \int_{-4}^{0} (x+4)\,dx$$

$$= -\left[\frac{x^2}{2} + 4x\right]_{-6}^{-4} + \left[\frac{x^2}{2} + 4x\right]_{-4}^{0}$$

$$= -\left[\frac{(-4)^2}{4} + 4(-4) - \left[\frac{(-6)^2}{2} + 4(-6)\right]\right] + \left[0 + 0 - \left[\frac{(-4)^2}{2} + 4(-4)\right]\right]$$

$$= -\left[\frac{16}{2} - 16 - \left[\frac{36}{2} - 24\right]\right] + \left[-\left(\frac{16}{2} - 16\right)\right]$$

$$= -[-8+6]+[8]$$

2 +8 =10 sq. unit

Question 18

Find the image of a point having position vector : $3\hat{i} - 2\hat{j} + \hat{k}$ in the plane $\vec{r}.\left(3\hat{i} - \hat{j} + 4\hat{k}\right) = 2.$

Ans. Let B be root of pont $A\left(3\hat{i} - 2\hat{j} + \hat{k}\right)$ in the plane $\bar{r}\left(3\hat{i} - \hat{j} + 4\hat{k}\right) = 2$ can of AB : is

$\bar{r}\left(3\hat{i} - 2\hat{j} + \hat{k}\right) + \lambda\left(3\hat{i} - \hat{j} + 4\hat{k}\right)$

$\therefore \quad \frac{x-3}{3} = \frac{y+2}{-1} = \frac{z-1}{4} = \lambda$

$\therefore \quad x = 3\lambda + 3, \; y = -\lambda - 2, \; z = 4\lambda + 1$

subtitute $x, y \,\&\, z$ in plane $3x - y + 4z = 2$

$\therefore \quad 3(3\lambda + 3) - (-\lambda - 2) + 4(4\lambda + 1) = 2$

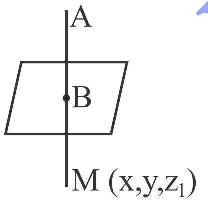
$9\lambda + 9 + \lambda + 2 + 16\lambda + 4 = 2$

$26\lambda + 13 = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{2}$

$\therefore \quad x = -\frac{3}{2} + 3, \; y = \frac{1}{2} - 2, \, z = -2 + 1$

$x = \frac{3}{2}, \; y = -\frac{3}{2}, \quad z = -1$

$\therefore$ by midpt formula,



$\frac{3}{2} = \frac{3 + x_1}{2}, \quad \frac{-3}{2} = \frac{-2 + y1}{z}, \quad -1 = \frac{1 - z_1}{2}$

$\boxed{x_1 = 0}, \quad \boxed{y_1 = -1}, \quad -z = 1 - z_1 \quad \boxed{z_1 = 1 + 2 = 3}$