5 Inverse Trigonometric Functions

5.1 INVERSE FUNCTIONS

To learn the concept of inverse function, consider the following examples :

(1) Let f be the function defined by $f(x) = 2x - 3$

i.e. $y = f(x) = 2x - 3$, $D_f = \mathbf{R}$ and $R_f = \mathbf{R}$.

On expressing x in terms of y, we get $x = \frac{y+3}{2}$.

We observe that for each y in $\mathbf{R}$ there exists a unique x in $\mathbf{R}$ such that $y = f(x) = 2x - 3$. The rule which associates to each $y \in R_f$, a unique $x \in D_f$ defines a new function (called 'f' inverse).

Note that the given function has the following two properties :

(*i*) No two elements of D_f are associated to the same element in R_f *i.e.* f is one-one.

(*ii*) For every $y \in R_f$, there exists a number x in D_f such that $f(x) = y$ *i.e* f is onto.

(2) Let f be the function defined by $f(x) = x^2$ *i.e.* $y = f(x) = x^2$.

$D_f = \mathbf{R}$ and R_f = the set of non- negative reals.

On expressing x in terms of y, we get $x = \pm\sqrt{y}$. ...(*i*)

We observe that corresponding to any $y > 0$, there are two values of x, therefore, (*i*) does not define x as a function of y. In fact, the function f is not one-one, and for this reason we cannot express x as a function of y.

From examples (1) and (2), we observe that only one-one functions have inverse functions. This leads to :

*Let f be a one-one function with domain D_f and range R_f then a function $g : R_f \to D_f$ defined by $g(y) = x$ where $f(x) = y$ is called **inverse** of f, denoted by f^{-1} (read as f inverse).*

*Thus, a function f is **inversible** (or **invertible**) iff f is one-one.*

Remarks

1. Let a real function $f : D_f \to R_f$, where D_f = domain of f and R_f = range of f, be invertible, then $f^{-1} : R_f \to D_f$ is given by $f^{-1}(y) = x$ iff $y = f(x)$ for all $x \in D_f$ and $y \in R_f$. Thus, the roles of x and y are just interchanged during the transition from f to f^{-1}

In fact, graph of f = $\{(x, y) : y = f(x), \text{ for all } x \in D_f\}$ and

graph of f^{-1} = $\{(y, x) : x = f^{-1}(y), \text{ for all } y \in R_f\}$.

Thus, $(x, y) \in$ graph of f iff $(y, x) \in$ graph of f^{-1}

The point (y, x) is the reflection of the point (x, y) in the line $y = x$, therefore, the graph of f^{-1} can be obtained from the graph of f by reflecting it through the line $y = x$.

2. It often happens that a given function f may not be one-one in the whole of its domain, but when we restrict it to a part of the domain, it may be so, *If a function is one-one on a part of its domain, it is said to be inversible on that part only.* If a function is inversible in several parts of its domain, it is said to have an inverse in each of these parts.

5.2 INVERSE TRIGONOMETRIC FUNCTIONS

1. Inverse sine function

Consider the sine function f defined by $f(x) = \sin x$, $D_f = \mathbf{R}$, $R_f = [-1, 1]$.

Table for the graph of f :

x	...	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{3}$	π	...
y	...	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	...

A portion of the graph is shown in fig. 5.1.

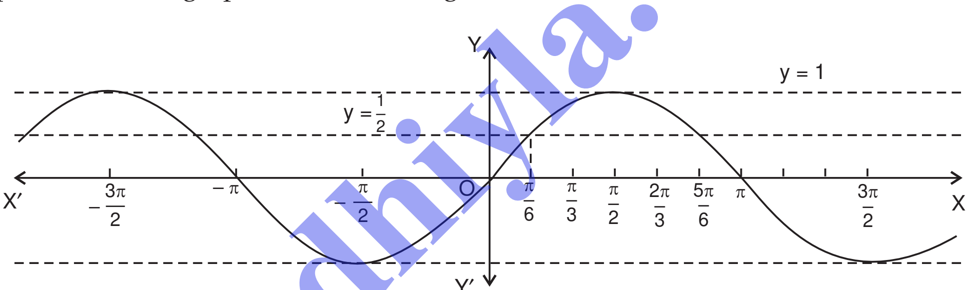

Fig. 5.1.

Note that the horizontal line $y = \frac{1}{2}$ meets its graph in many points, so f *is not one-one.*

But if we restrict the domain from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, both inclusive, we observe that in this part of the domain f is one-one. Therefore, the function $y = f(x) = \sin x$ with $D_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $R_f = [-1, 1]$ has an inverse function called the **inverse sine function** or the **arc sine function,** denoted by $\sin^{-1}$. Thus,

$$\boldsymbol{y = \sin^{-1} x \text{ iff } x = \sin y \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].}$$

It has the following properties :

(*i*) Domain of $\sin^{-1} x$ is $[-1, 1]$

and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(*ii*) $\sin(\sin^{-1} x) = x$, for

$x \in [-1, 1]$ *i.e.* $|x| \le 1$.

(*iii*) $\sin^{-1}(\sin y) = y$, for

$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *i.e.* $|y| \le \frac{\pi}{2}$.

(*iv*) $\sin^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

is strictly increasing and is one-one.

The graph of $\sin^{-1} x$ is shown in fig. 5.2.

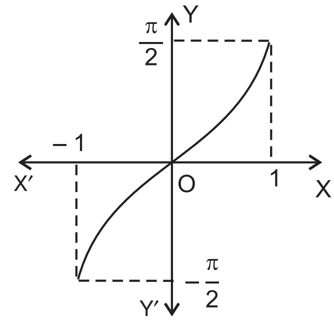

Fig. 5.2.

Remarks

1. Besides $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, there exist other intervals where the sine function is one-one and hence has an inverse function but here $\sin^{-1} x$ shall always mean the function

$$\sin^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

defined above (unless stated otherwise). The portion of the curve for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ is known as the **principal value branch** of the function $y = \sin^{-1} x$ and these values of y are known as the **principal values** of the function $y = \sin^{-1} x$.

2. The graph of $\sin^{-1}$ function can be obtained from the graph of the original function by interchanging the roles of x and y *i.e.* if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse sine function. The graph of $\sin^{-1}$ function is the mirror image along the line $y = x$ of the corresponding original function. This can be visualised by looking the graphs of $y = \sin x$ and $y = \sin^{-1} x$ in the same axes as shown in fig. 5.3.

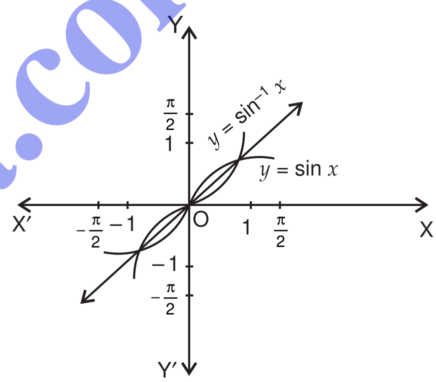

Fig. 5.3.

3. It may be noted that besides $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, there exist other intervals such as $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$, $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$ etc. where the sine function is one-one and hence has an inverse function. Thus, the range of other branches are $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$, $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$ etc.

2. Inverse cosine function

Consider the cosine function f defined by

$$f(x) = \cos x,\ D_f = \mathbf{R} \text{ and } R_f = [-1, 1].$$

A portion of the graph of cos x is shown in fig. 5.4.

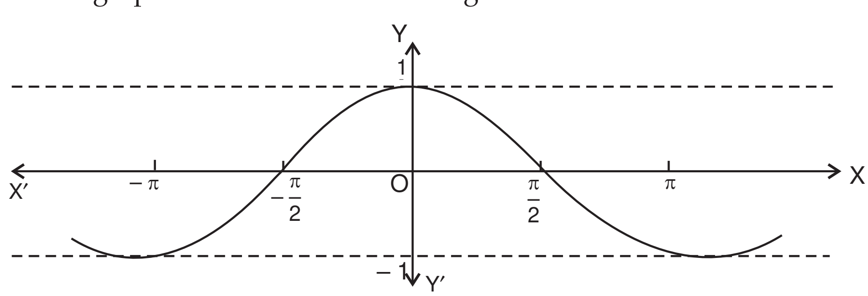

Fig. 5.4.

Clearly, f *is not one-one* but if we restrict the domain to $[0, \pi]$, f is one-one and so it has an inverse function called the **inverse cosine function** or the **arc cosine function**, denoted by $\cos^{-1}$. Thus,

$y = \cos^{-1} x$ iff $x = \cos y$ and $y \in [0, \pi]$.

It has the following properties :

(*i*) Domain of $\cos^{-1} x$ is $[-1, 1]$ and its range is $[0, \pi]$.

(*ii*) $\cos(\cos^{-1} x) = x$, for $x \in [-1, 1]$ *i.e.* $|x| \le 1$.

(*iii*) $\cos^{-1}(\cos y) = y$, for $y \in [0, \pi]$.

(*iv*) $\cos^{-1} : [-1, 1] \to [0, \pi]$ is strictly decreasing and is one-one.

The graph of $\cos^{-1} x$ is shown in fig. 5.5.

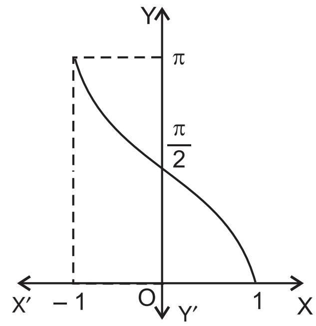

Fig. 5.5.

The portion of the curve for which $0 \le y \le \pi$ is known as the **principal value branch** of the function $y = \cos^{-1} x$ and these values of y are known as the **principal values** of the function $y = \cos^{-1} x$.

Remark. Note that the range of other branches of $\cos^{-1} x$ are $[\pi, 2\pi]$, $[2\pi, 3\pi]$, $[-\pi, 0]$ etc.

3. Inverse tangent function

Consider the tangent function f defined by $f(x) = \tan x$, $D_f = \mathbf{R}$ except odd multiples of $\frac{\pi}{2}$ and $R_f = \mathbf{R}$.

A portion of the graph of tan x is shown in fig. 5.6.

Clearly, f *is not one-one* but if we restrict the domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, f is one-one and so it has an inverse function called the **inverse tangent function** or the **arc tangent function,** denoted by $\tan^{-1}$. Thus,

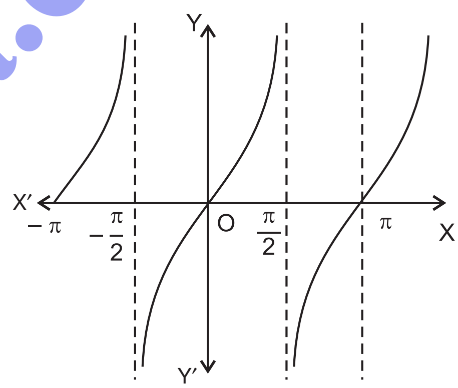

Fig. 5.6.

$y = \tan^{-1} x$ iff $x = \tan y$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

It has the following properties :

(*i*) Domain of $\tan^{-1} x$ is $\mathbf{R}$ and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(*ii*) $\tan(\tan^{-1} x) = x$, $x \in \mathbf{R}$.

(*iii*) $\tan^{-1}(\tan y) = y$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(*iv*) $\tan^{-1} : \mathbf{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is strictly increasing and is one-one.

A portion of the graph of $\tan^{-1} x$ is shown in fig. 5.7.

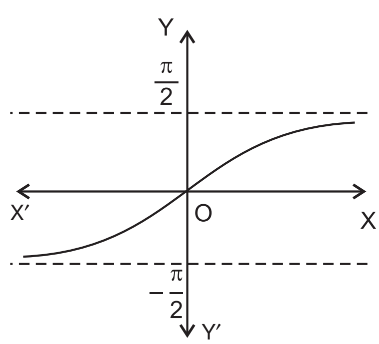

Fig. 5.7.

The portion of the curve for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$ is known as the **principal value branch** of the function $y = \tan^{-1} x$ and these values of y are known as the **principal values.**

4. Inverse cotangent function

Consider the cotangent function f defined by $f(x) = \cot x$, $D_f = \mathbf{R}$ except even multiples of $\frac{\pi}{2}$ and $R_f = \mathbf{R}$.

A portion of the graph of cot x is shown is fig. 5.8.

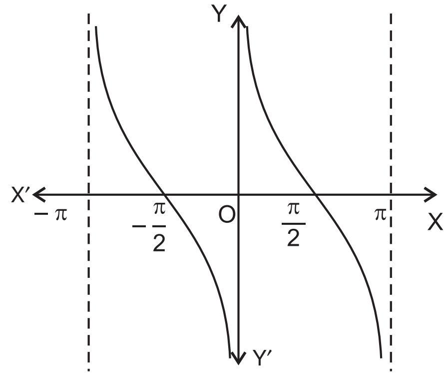

Fig. 5.8.

Clearly, f is *not one-one* but if we restrict the domain to $(0, \pi)$, f is one-one and so it has an inverse function called **inverse cotangent function** or **arc cotangent function,** denoted by $\cot^{-1}$. Thus,

$y = \cot^{-1} x$ iff $x = \cot y$ and $y \in (0, \pi)$.

It has the following properties :

(*i*) Domain of $\cot^{-1} x$ is **R** and its range is $(0, \pi)$.

(*ii*) $\cot(\cot^{-1} x) = x$, $x \in \mathbf{R}$.

(*iii*) $\cot^{-1}(\cot y) = y$, $y \in (0, \pi)$.

(*iv*) $\cot^{-1} : \mathbf{R} \to (0, \pi)$ is strictly decreasing and is one-one.

A portion of the graph of $\cot^{-1} x$ is shown in fig. 5.9.

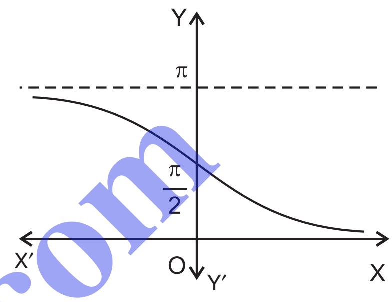

Fig. 5.9.

The portion of the curve for which $0 < y < \pi$ is known as the **principal value branch** of the function $y = \cot^{-1} x$ and these values of y are known as the **principal values**.

5. Inverse secant function

Consider the function f defined by $f(x) = \sec x$,

$D_f = \mathbf{R}$ except odd multiples of $\frac{\pi}{2}$ and $R_f = (-\infty, -1] \cup [1, \infty)$.

A portion of the graph of sec x is shown in fig. 5.10.

Clearly, f is *not one-one* but if we restrict the domain to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, f is one-one and so it has an inverse function called **inverse secant function** or **arc secant function,** denoted by $\sec^{-1}$. Thus,

$y = \sec^{-1} x$ iff $x = \sec y$ and

$y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

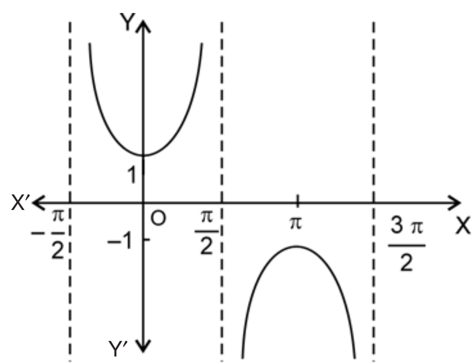

Fig. 5.10.

It has the following properties :

(*i*) Domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$ and its range is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

(*ii*) $\sec(\sec^{-1} x) = x$, for $|x| \geq 1$.

(*iii*) $\sec^{-1}(\sec y) = y$, $y \in (0, \pi)$, $y \neq \frac{\pi}{2}$.

(*iv*) $\sec^{-1} x$ is strictly increasing (piecewise) and is one-one.

The portion of the curve for which $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$, is known as the **principal value branch** of the function $y = \sec^{-1} x$ and these values of y are called the **principal values.**

6. Inverse cosecant function

Consider the function f defined by

$f(x) = \text{cosec } x$, $D_f = \mathbf{R}$ except even multiples of $\frac{\pi}{2}$ and $R_f = (-\infty, -1] \cup [1, \infty)$.

A portion of the graph of cosec x is shown in fig. 5.11.

Clearly, f is *not one-one* but if we restrict the domain to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, f is one-one and so it has an inverse function called **inverse cosecant function** or **arc cosecant function,** denoted by cosec^{-1}. Thus,

$y = \text{cosec}^{-1} x$ iff $x = \text{cosec } y$,

$y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

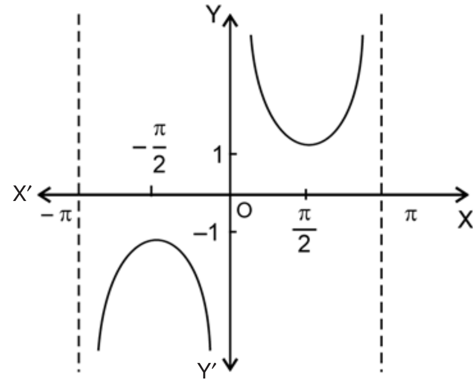

Fig. 5.11.

It has the following properties :

(*i*) Domain of $\text{cosec}^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$ and its range is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

(*ii*) $\text{cosec}(\text{cosec}^{-1} x) = x$, for $|x| \geq 1$.

(*iii*) $\text{cosec}^{-1}(\text{cosec } y) = y$; $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y \neq 0$.

(*iv*) $\text{cosec}^{-1} x$ is strictly decreasing (piecewise) and is one-one.

The portion of the curve for which $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$, is known as the **principal value branch** of the function $y = \text{cosec}^{-1} x$ and these values of y are known as **principal values.**

5.3 SOME IMPORTANT RESULTS

1. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $|x| \leq 1$.

2. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $x \in \mathbf{R}$.

3. $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$, $|x| \geq 1$.

4. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $xy < 1$.

5. $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, $xy > -1$.

6. (*i*) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$, $0 \leq x \leq 1$

(*ii*) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$, $0 \leq x \leq 1$.

7. (*i*) $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1} x$, $|x| < 1$

(*ii*) $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cos^{-1} x$, $0 < x \leq 1$.

8. $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1} x$, $|x| \leq 1$.

9. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x$, $x \geq 0$.

10. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x$, $|x| < 1$.

11. $\text{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$.

12. $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$.

13. $\cot^{-1} x = \begin{cases} \tan^{-1}\dfrac{1}{x}, & x > 0 \\ \pi + \tan^{-1}\dfrac{1}{x}, & x < 0 \end{cases}$

14. $\cos(\sin^{-1} x) = \sin(\cos^{-1} x) = \sqrt{1-x^2}, \ |x| \le 1.$

15. (*i*) $\sin^{-1}(-x) = -\sin^{-1} x, \ |x| \le 1$ (*ii*) $\cos^{-1}(-x) = \pi - \cos^{-1} x, \ |x| \le 1$

(*iii*) $\tan^{-1}(-x) = -\tan^{-1} x, \ x \in \mathbf{R}$ (*iv*) $\cot^{-1}(-x) = \pi - \cot^{-1} x, \ x \in \mathbf{R}$

(*v*) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, \ |x| \ge 1$ (*vi*) $\sec^{-1}(-x) = \pi - \sec^{-1} x, \ |x| \ge 1.$

16. For suitable values of x and y

(*i*) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

(*ii*) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

(*iii*) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

(*iv*) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right).$

Proof 1. Let $\sin^{-1} x = y \Rightarrow x = \sin y \Rightarrow x = \cos\left(\frac{\pi}{2} - y\right)$

$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - y \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$

$\Rightarrow \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \ |x| \le 1.$

(It may be noted that when $y = \sin^{-1} x$ and y is a principal value then $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} \ge -y \ge -\frac{\pi}{2} \Rightarrow \frac{\pi}{2} + \frac{\pi}{2} \ge \frac{\pi}{2} - y \ge \frac{\pi}{2} - \frac{\pi}{2}$

$\Rightarrow \pi \ge \frac{\pi}{2} - y \ge 0 \Rightarrow 0 \le \frac{\pi}{2} - y \le \pi$

$\Rightarrow \frac{\pi}{2} - y$ is a principal value of $\cos^{-1} x$.)

2. Let $\tan^{-1} x = y \Rightarrow x = \tan y \Rightarrow x = \cot\left(\frac{\pi}{2} - y\right)$

$\Rightarrow \cot^{-1} x = \frac{\pi}{2} - y \Rightarrow \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$

$\Rightarrow \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$

3. Left as an exercise for the reader.

4. Let $\tan^{-1} x = \alpha$ and $\tan^{-1} y = \beta \Rightarrow x = \tan\alpha$ and $y = \tan\beta$.

Now $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{x+y}{1-xy}$

$\Rightarrow \alpha + \beta = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right).$

(It may be noted that when $\alpha = \tan^{-1} x$, $\beta = \tan^{-1} y$, then

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}, \ -\frac{\pi}{2} < \beta < \frac{\pi}{2}.$$

Given $xy < 1 \Rightarrow \tan\alpha\tan\beta < 1 \Rightarrow \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} < 1$

But $\cos\alpha\cos\beta > 0$ because $-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \Rightarrow \cos\alpha > 0$ and $-\frac{\pi}{2} < \beta < \frac{\pi}{2} \Rightarrow \cos\beta > 0$.

$\therefore\ \sin\alpha\sin\beta < \cos\alpha\cos\beta \Rightarrow 0 < \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\Rightarrow\ 0 < \cos(\alpha+\beta) \Rightarrow \cos(\alpha+\beta) > 0 \ \Rightarrow\ -\frac{\pi}{2} < \alpha+\beta < \frac{\pi}{2}\Big)$

Remark. If $xy > 1$, then $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\frac{x+y}{1-xy}$.

5. Left as an exercise for the reader.

6. (*i*) Let $\sin^{-1}x = y \Rightarrow x = \sin y$; as $0 \le x \le 1$, $0 \le y \le \frac{\pi}{2}$

$\therefore\quad \sqrt{1-x^2} = \sqrt{1-\sin^2 y} = \sqrt{\cos^2 y} = |\cos y| = \cos y$

$(\because\ 0 \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0 \Rightarrow |\cos y| = \cos y)$

$\Rightarrow\quad y = \cos^{-1}\sqrt{1-x^2}$ $\quad(\because\ 0 \le y \le \frac{\pi}{2})$

$\Rightarrow\quad \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$.

(*ii*) Let $\cos^{-1}x = y \Rightarrow x = \cos y$; as $0 \le x \le 1$, $0 \le y \le \frac{\pi}{2}$

$\therefore\quad \sqrt{1-x^2} = \sqrt{1-\cos^2 y} = \sqrt{\sin^2 y} = |\sin y| = \sin y$

$(\because\ 0 \le y \le \frac{\pi}{2} \Rightarrow \sin y \ge 0 \Rightarrow |\sin y| = \sin y)$

$\Rightarrow\quad y = \sin^{-1}\sqrt{1-x^2}$ $\quad(\because\ 0 \le y \le \frac{\pi}{2})$

$\Rightarrow\quad \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$.

7. (*i*) Let $x = \sin y \Rightarrow y = \sin^{-1}x$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$ $\quad(\because\ |x| < 1)$

$\therefore\quad \sqrt{1-x^2} = \sqrt{1-\sin^2 y} = \sqrt{\cos^2 y} = |\cos y| = \cos y$ $\quad\left(\because -\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow \cos y > 0\right)$

$\therefore\quad \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sin y}{\cos y}\right) = \tan^{-1}(\tan y) = y$

$\Rightarrow\quad \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1}x$.

(*ii*) Let $x = \cos y \Rightarrow y = \cos^{-1}x$, $0 \le y < \frac{\pi}{2}$ $\quad(\because\ 0 < x \le 1)$

$\therefore\quad \sqrt{1-x^2} = \sqrt{1-\cos^2 y} = \sqrt{\sin^2 y} = |\sin y| = \sin y$ $\quad(\because\ 0 \le y < \frac{\pi}{2} \Rightarrow \sin y \ge 0)$

$\therefore\quad \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1}\left(\frac{\sin y}{\cos y}\right) = \tan^{-1}(\tan y) = y$

$\Rightarrow\quad \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cos^{-1}x$.

8. Let $\tan^{-1}x = y \Rightarrow x = \tan y$; as $|x| \le 1$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$.

Now $\sin 2y = \dfrac{2\tan y}{1-\tan^2 y} = \dfrac{2x}{1-x^2} \Rightarrow 2y = \sin^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$\left(\because -\dfrac{\pi}{4} \le y \le \dfrac{\pi}{4} \Rightarrow -\dfrac{\pi}{2} \le 2y \le \dfrac{\pi}{2}\right)$

$\Rightarrow \quad 2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

9. Left as an exercise for the reader.

10. On taking $y = x$ in result 4, we get

$\tan^{-1} x + \tan^{-1} x = \tan^{-1} \dfrac{x+x}{1-x.x}$

$\Rightarrow 2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

11. Let $\text{cosec}^{-1} x = y, -\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}, y \ne 0$

$\Rightarrow \quad x = \text{cosec } y \Rightarrow \dfrac{1}{x} = \dfrac{1}{\text{cosec } y} \Rightarrow \dfrac{1}{x} = \sin y$

$\Rightarrow \quad y = \sin^{-1} \dfrac{1}{x}$ $\quad \left(\because -\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}, y \ne 0\right)$

$\Rightarrow \quad \text{cosec}^{-1} x = \sin^{-1} \dfrac{1}{x}, \ |x| \ge 1.$

12. Let $\sec^{-1} x = y, 0 \le y \le \pi, y \ne \dfrac{\pi}{2}$

$\Rightarrow \quad x = \sec y \Rightarrow \dfrac{1}{x} = \dfrac{1}{\sec y} \Rightarrow \dfrac{1}{x} = \cos y$

$\Rightarrow \quad y = \cos^{-1} \dfrac{1}{x}$ $\quad \left(\because 0 \le y \le \pi, y \ne \dfrac{\pi}{2}\right)$

$\Rightarrow \quad \sec^{-1} x = \cos^{-1} \dfrac{1}{x}, \ |x| \ge 1.$

13. ***When x > 0***

Let $\cot^{-1} x = y$, as $x > 0, 0 < y < \dfrac{\pi}{2}$

$\Rightarrow \quad x = \cot y \Rightarrow \dfrac{1}{x} = \dfrac{1}{\cot y} \Rightarrow \dfrac{1}{x} = \tan y$

$\Rightarrow \quad y = \tan^{-1} \dfrac{1}{x}$ $\quad \left(\because 0 < y < \dfrac{\pi}{2}\right)$

$\Rightarrow \quad \cot^{-1} x = \tan^{-1} \dfrac{1}{x}, x > 0.$

When x < 0

Let $\cot^{-1} x = y$, as $x < 0, \dfrac{\pi}{2} < y < \pi$.

Now $\dfrac{\pi}{2} < y < \pi \Rightarrow -\dfrac{\pi}{2} < y - \pi < 0.$

$\cot^{-1} x = y \Rightarrow x = \cot y \Rightarrow \dfrac{1}{x} = \dfrac{1}{\cot y}$

$\Rightarrow \quad \dfrac{1}{x} = \tan y = -\tan(\pi - y) = \tan(y - \pi)$

$\Rightarrow \quad y - \pi = \tan^{-1} \dfrac{1}{x}$ $\quad \left(\because -\dfrac{\pi}{2} < y - \pi < 0\right)$

$\Rightarrow \quad \cot^{-1} x = \pi + \tan^{-1} \dfrac{1}{x}, x < 0.$

14. Let $\sin^{-1} x = y, -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow x = \sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Clearly for this value of y, $\cos y \ge 0$,

$\therefore \quad \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$\Rightarrow \quad \cos(\sin^{-1} x) = \sqrt{1-x^2}$ ...(*i*)

Now, let $\cos^{-1} x = t, 0 \le t \le \pi$

$\Rightarrow \quad x = \cos t, 0 \le t \le \pi$.

Clearly, for this value of t, $\sin t \ge 0$,

$\therefore \quad \sin t = \sqrt{1-\cos^2 t} = \sqrt{1-x^2}$

$\Rightarrow \quad \sin(\cos^{-1} x) = \sqrt{1-x^2}$ ...(*ii*)

From (*i*) and (*ii*), we get

$\cos(\sin^{-1} x) = \sin(\cos^{-1} x) = \sqrt{1-x^2}, \ |x| \le 1.$

15. (*i*) Let $\sin^{-1} x = y \Rightarrow x = \sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow -x = -\sin y, \frac{\pi}{2} \ge -y \ge -\frac{\pi}{2}$

$\Rightarrow -x = \sin(-y), -\frac{\pi}{2} \le -y \le \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(-x) = -y$

$\Rightarrow \sin^{-1}(-x) = -\sin^{-1} x, \ |x| \le 1.$

(*ii*) Let $\cos^{-1} x = y \Rightarrow x = \cos y, 0 \le y \le \pi$

$\Rightarrow -x = -\cos y, 0 \ge -y \ge -\pi$ *i.e.* $\pi \ge \pi - y \ge 0$

$\Rightarrow -x = \cos(\pi - y), 0 \le \pi - y \le \pi$

$\Rightarrow \cos^{-1}(-x) = \pi - y$

$\Rightarrow \cos^{-1}(-x) = \pi - \cos^{-1} x, |x| \le 1.$

(*iii*) The proofs of the other parts are left as exercises for the reader.

16. (*i*) Let $\sin^{-1} x = \alpha$ and $\sin^{-1} y = \beta$

$\Rightarrow x = \sin\alpha$ and $y = \sin\beta$

$\therefore \quad \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$

and $\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-y^2}$.

Now, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= x\sqrt{1-y^2} + \sqrt{1-x^2}\ y$

$\Rightarrow \alpha + \beta = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$\Rightarrow \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$.

Note. Here $|x| \le 1, |y| \le 1$ if $xy \le 0$ or if $xy > 0$ and $x^2 + y^2 \le 1$.

(*ii*) Left as an exercise for the reader.

(*iii*) Let $\cos^{-1} x = \alpha$ and $\cos^{-1} y = \beta$

$\Rightarrow x = \cos\alpha$ and $y = \cos\beta$.

$\therefore \quad \sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-x^2}$

and $\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-y^2}$.

Now, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= xy - \sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow \alpha + \beta = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\Rightarrow \cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right).$

(*iv*) Left as an exercise for the reader.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the principal values of :*

(*i*) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (*ii*) $\cot^{-1}(\sqrt{3})$ (*iii*) $\text{cosec}^{-1}(\sqrt{2})$.

Solution. (*i*) Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = x,\ 0 \le x \le \pi$

$\Rightarrow \quad \cos x = \frac{\sqrt{3}}{2} \Rightarrow \cos x = \cos\frac{\pi}{6}$

$\Rightarrow \quad x = \frac{\pi}{6} \Rightarrow \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$

(*ii*) Let $\cot^{-1}(\sqrt{3}) = x,\ 0 < x < \pi$

$\Rightarrow \quad \cot x = \sqrt{3} \Rightarrow \cot x = \cot\frac{\pi}{6}$

$\Rightarrow \quad x = \frac{\pi}{6} \Rightarrow \cot^{-1}(\sqrt{3}) = \frac{\pi}{6}.$

(*iii*) Let $\text{cosec}^{-1}(\sqrt{2}) = x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2},\ x \ne 0$

$\Rightarrow \quad \text{cosec}\, x = \sqrt{2} \Rightarrow \text{cosec}\, x = \text{cosec}\frac{\pi}{4}$

$\Rightarrow \quad x = \frac{\pi}{4} \Rightarrow \text{cosec}^{-1}(\sqrt{2}) = \frac{\pi}{4}.$

Example 2. *Find the principal values of :*

(*i*) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (*ii*) $\sec^{-1}(-2)$ (*iii*) $\cot^{-1}(-1)$.

Solution. (*i*) Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\Rightarrow \quad \sin x = -\frac{\sqrt{3}}{2} \Rightarrow \sin x = -\sin\frac{\pi}{3} \Rightarrow \sin x = \sin\left(-\frac{\pi}{3}\right)$

$\Rightarrow \quad x = -\frac{\pi}{3} \Rightarrow \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}.$

Alternative method

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \qquad (\because \sin^{-1}(-x) = -\sin^{-1} x)$

Let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\Rightarrow \quad \sin x = \frac{\sqrt{3}}{2} \Rightarrow \sin x = \sin\frac{\pi}{3}$

$\Rightarrow \quad x = \frac{\pi}{3} \Rightarrow \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$

$\therefore \quad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}.$

(*ii*) $\sec^{-1}(-2) = \pi - \sec^{-1} 2$ $\quad (\because \sec^{-1}(-x) = \pi - \sec^{-1} x)$

Let $\sec^{-1} 2 = x,\ 0 \le x \le \pi,\ x \ne \frac{\pi}{2}$

$\Rightarrow \quad \sec x = 2 \Rightarrow \sec x = \sec \frac{\pi}{3}$

$\Rightarrow \quad x = \frac{\pi}{3} \Rightarrow \sec^{-1} x = \frac{\pi}{3}.$

$\therefore \quad \sec^{-1}(-2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$

(*iii*) $\cot^{-1}(-1) = -\cot^{-1} 1$ $\quad (\because \cot^{-1}(-x) = \pi - \cot^{-1} x)$

Let $\cot^{-1} 1 = x,\ 0 < x < \pi$

$\Rightarrow \quad \cot x = 1 \Rightarrow \cot x = \cot \frac{\pi}{4}$

$\Rightarrow \quad x = \frac{\pi}{4} \Rightarrow \cot^{-1} 1 = \frac{\pi}{4}.$

$\therefore \quad \cot^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$

Example 3. *Find the principal values of :*

(*i*) $\sin^{-1}\left(\sin \frac{3\pi}{5}\right)$ $\qquad$ (*ii*) $\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$ $\qquad$ (*iii*) $\tan^{-1}\left(\tan \frac{6\pi}{7}\right).$

Solution. (*i*) Let $\sin^{-1}\left(\sin \frac{3\pi}{5}\right) = y,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \quad \sin y = \sin \frac{3\pi}{5} \Rightarrow \sin y = \sin\left(\pi - \frac{2\pi}{5}\right)$

$\Rightarrow \quad \sin y = \sin \frac{2\pi}{5},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{2\pi}{5} \Rightarrow \sin^{-1}\left(\sin \frac{3\pi}{5}\right) = \frac{2\pi}{5}.$

(*ii*) Let $\cos^{-1}\left(\cos \frac{7\pi}{6}\right) = y,\ 0 \le y \le \pi$

$\Rightarrow \quad \cos y = \cos \frac{7\pi}{6} \Rightarrow \cos y = \cos\left(2\pi - \frac{5\pi}{6}\right)$

$\Rightarrow \quad \cos y = \cos \frac{5\pi}{6},\ 0 \le y \le \pi$

$\Rightarrow \quad y = \frac{5\pi}{6} \Rightarrow \cos^{-1}\left(\cos \frac{7\pi}{6}\right) = \frac{5\pi}{6}.$

(*iii*) Let $\tan^{-1}\left(\tan \frac{6\pi}{7}\right) = y,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow \quad \tan y = \tan \frac{6\pi}{7} \Rightarrow \tan y = \tan\left(\pi - \frac{\pi}{7}\right)$

$\Rightarrow \quad \tan y = -\tan \frac{\pi}{7} = \tan\left(-\frac{\pi}{7}\right),\ -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow \quad y = -\frac{\pi}{7} \Rightarrow \tan^{-1}\left(\tan \frac{6\pi}{7}\right) = -\frac{\pi}{7}.$

Example 4. *Show that* $\sin^{-1} \frac{\sqrt{3}}{2} + 2 \tan^{-1} \frac{1}{\sqrt{3}} = \frac{2\pi}{3}.$

Solution. Let $\sin^{-1} \frac{\sqrt{3}}{2} = \alpha,\ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

$\Rightarrow \quad \sin \alpha = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{3}.$

Example 33. *If* $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, *prove that :*

(*i*) $x^2 - y^2 - z^2 + 2yz\sqrt{1-x^2} = 0$ *(I.S.C. 2009)*

(*ii*) $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$.

Solution. (*i*) Given $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} z$

$\Rightarrow \quad \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \pi - \sin^{-1} z$

$\Rightarrow \quad x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin(\pi - \sin^{-1} z)$

$\Rightarrow \quad x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin(\sin^{-1} z)$

$\Rightarrow \quad x\sqrt{1-y^2} + y\sqrt{1-x^2} = z$

$\Rightarrow \quad x\sqrt{1-y^2} = z - y\sqrt{1-x^2}$.

Squaring both sides, we get

$$x^2(1-y^2) = z^2 + y^2(1-x^2) - 2yz\sqrt{1-x^2}$$

$\Rightarrow \quad x^2 - x^2y^2 = z^2 + y^2 - x^2y^2 - 2yz\sqrt{1-x^2}$

$\Rightarrow \quad x^2 - y^2 - z^2 + 2yz\sqrt{1-x^2} = 0.$

(*ii*) From part (*i*), we get $x^2 - y^2 - z^2 = -2yz\sqrt{1-x^2}$.

Squaring both sides, we get

$$(x^2 - y^2 - z^2)^2 = 4y^2z^2(1-x^2)$$

$\Rightarrow \quad x^4 + y^4 + z^4 - 2x^2y^2 - 2z^2x^2 + 2y^2z^2 = 4y^2z^2 - 4x^2y^2z^2$

$\Rightarrow \quad x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$.

EXERCISE 5.1

1. Find the principal values of :

(*i*) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (*ii*) $\cos^{-1}\left(-\frac{1}{2}\right)$ (*iii*) $\cot^{-1}(-\sqrt{3})$

(*iv*) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (*v*) $\text{cosec}^{-1}(-2)$ (*vi*) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

2. Evaluate the following :

(*i*) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$ (*ii*) $\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$ (*iii*) $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

(*iv*) $\cot(\tan^{-1}\sqrt{3})$ (*v*) $\sin\left(\frac{\pi}{6} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ (*vi*) $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right)$.

3. Show that :

(*i*) $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) \neq \frac{5\pi}{6}$. What is its value?

(*ii*) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right) \neq -\frac{\pi}{6}$. What is its value?

(*iii*) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$. What is its value?

4. Using principal values, prove the following :

(*i*) $\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{2\pi}{3}$

(*ii*) $\sin^{-1}\frac{1}{\sqrt{2}}-3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{3\pi}{4}$

(*iii*) $\tan^{-1}(-1)+\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\frac{\pi}{2}$

(*iv*) $\operatorname{cosec}^{-1}(-1)+\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$.

5. Evaluate the following :

(*i*) $\tan\left(\cos^{-1}\frac{8}{17}\right)$ (*ii*) $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$ (*iii*) $\operatorname{cosec}\left(\cos^{-1}\left(-\frac{12}{13}\right)\right)$.

6. Evaluate the following :

(*i*) $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$ (*ii*) $\tan^{-1}\left(2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right)$.

7. Prove the following :

(*i*) $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x,\ x<1$

(*ii*) $\operatorname{cosec}^{-1}\frac{1}{x}=\sec^{-1}\frac{1}{\sqrt{1-x^2}}=\cot^{-1}\frac{\sqrt{1-x^2}}{x}$.

8. Prove that $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$.

9. Prove the following (for suitable values of x, y) :

(*i*) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$ (*ii*) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

(*iii*) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

(*iv*) $\tan^{-1}\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}=\tan^{-1}\sqrt{x}+\tan^{-1}\sqrt{y}$

(*v*) $\tan^{-1}\frac{x+\sqrt{x}}{1-x^{3/2}}=\tan^{-1}x+\tan^{-1}\sqrt{x}$ (*vi*) $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$.

Hint. (*vi*) Put $\sqrt{x}=\tan y$.

10. Write the following functions in simplest form :

(*i*) $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ (*ii*) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$

(*iii*) $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$ (*iv*) $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$

(*v*) $\tan^{-1}\left(\frac{2\sqrt{x}}{1-x}\right)$ (*vi*) $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

(*vii*) $\sin^{-1}\left(\sqrt{\frac{x}{1+x}}\right)$ (*viii*) $\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$

(*ix*) $\cos^{-1}(2x^2-1)$ (*x*) $\cos^{-1}(1-2x^2)$

(*xi*) $\cos^{-1}\left(\sqrt{1-x^2}\right)$ (*xii*) $\tan^{-1}\left(\frac{3a^2x-x^3}{a(a^2-3x^2)}\right)$

(*xiii*) $\cot^{-1}\left(\sqrt{1+x^2}-x\right)$ (*xiv*) $\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$.

Hint. (*vii*) Put $\sqrt{x}=\tan y$.

ANSWERS

EXERCISE 5.1

1. (*i*) $\frac{\pi}{4}$ (*ii*) $\frac{2\pi}{3}$ (*iii*) $\frac{5\pi}{6}$ (*iv*) $-\frac{\pi}{6}$ (*v*) $-\frac{\pi}{6}$ (*vi*) $\frac{\pi}{6}$.

2. (*i*) $\frac{\pi}{6}$ (*ii*) $-\frac{\pi}{4}$ (*iii*) $-\frac{\pi}{4}$ (*iv*) $\frac{1}{\sqrt{3}}$ (*v*) 1 (*vi*) -1.

3. (*i*) $-\frac{\pi}{6}$ (*ii*) $\frac{\pi}{6}$ (*iii*) $\frac{\pi}{3}$. **5.** (*i*) $\frac{15}{8}$ (*ii*) $\frac{4}{5}$ (*iii*) $\frac{13}{5}$.

6. (*i*) $\frac{1}{\sqrt{10}}$ (*ii*) $\frac{\pi}{4}$.

10. (*i*) $\frac{x}{2}$ (*ii*) $\frac{\pi}{4} - x$ (*iii*) $\frac{x}{2}$ (*iv*) $\frac{\pi}{2} - \sec^{-1} x$ (*v*) $2 \tan^{-1} \sqrt{x}$

(*vi*) $2 \cos^{-1} x$ (*vii*) $\tan^{-1} \sqrt{x}$ (*viii*) $\frac{1}{2} \cos^{-1} x$ (*ix*) $2 \cos^{-1} x$ (*x*) $2 \sin^{-1} x$

(*xi*) $\sin^{-1} x$ (*xii*) $3 \tan^{-1} \frac{x}{a}$ (*xiii*) $\frac{\pi}{4} + \frac{1}{2} \tan^{-1} x$ (*xiv*) $\sin^{-1} x + \sin^{-1} y$.

11. (*i*) $\frac{x+y}{1-xy}$. **16.** (*i*) $\frac{17}{6}$ (*ii*) $\frac{33}{65}$.

18. (*i*) $\frac{1}{4}$ (*ii*) $\frac{1}{3}$ (*iii*) $0, \pm\frac{1}{2}$ (*iv*) 17 (*v*) $-\frac{1}{12}$.

19. (*i*) $\pm \frac{4\sqrt{5}}{9}$ (*ii*) $\frac{a+b}{1-ab}$.

EXERCISE 5.2

2. $\frac{\pi}{3}$. **3.** (*i*) $\frac{\sqrt{5}}{3}$ (*ii*) $-\frac{1}{2}$. **4.** $\frac{4}{3}$. **5.** $n\pi, n\pi + \frac{\pi}{4}$, where $n \in$ **I.**

6. $x = \frac{\sqrt{3}}{2}, y = 0$. **7.** $x = 1, y = 2; x = 2, y = 7$.

CHAPTER TEST

1. $-\frac{\sqrt{24}}{5}$. **6.** (*i*) $\frac{\sqrt{3}}{2}$ (*ii*) $\pm\frac{2}{3}$ (*iii*) $\frac{1}{2}$ (*iv*) $1, -\frac{1}{6}$ (*v*) $0, \pm\frac{1}{2}$ (*vi*) 2.

8. $x = ab$. **9.** $\frac{\pi}{4}$.