

3

Boolean Algebra

3.1 BOOLEAN ALGEBRA

In 1854, George Boole introduced the following formalism which eventually became Boolean Algebra.

Definition. An algebraic system consisting of a set B of elements $a, b, c \dots$ and two binary operations called the **sum** and **product**, denoted respectively by $+$ and $\cdot$, is called a **boolean algebra** iff for all $a, b, c \in B$, the following axioms are satisfied :

- (1) $a + b, a \cdot b \in B$ (closure property)
- (2) $a + b = b + a$ and $a \cdot b = b \cdot a$ (commutative property)
- (3) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative property)
- (4) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $a + (b \cdot c) = (a + b) \cdot (a + c)$ (distributive laws)
- (5) An additive identity 0 and a multiplicative identity 1 (both belonging to B) exist such that
 $\forall a \in B, a + 0 = a$ and $a \cdot 1 = a$ (identity)
- (6) for every $a \in B$, there exists an element $a' \in B$ such that
 $a + a' = 1$ and $a \cdot a' = 0$ (complement or inverse)

Remarks

1. Some authors include another axiom—that of *cardinality*—that there are atleast two elements a and b in B such that $a \neq b$.
2. Some axioms, especially (6) are quite different from usual arithmetic or algebraic structures. First, the additive and multiplicative inverses of an element are usually different, e.g., in real numbers, additive inverse of 2 is (-2) whereas multiplicative inverse of 2 is $\frac{1}{2}$. In boolean algebra, both inverses are the same. Secondly, in a usual algebra, we have $a + a' = 0$ and $a \cdot a' = 1$. In boolean algebra, we have $a + a' = 1$ and $a \cdot a' = 0$. This is more like sets, where $A \cup A' = \xi$ (universal set) and $A \cap A' = \phi$ (null set). Observe that ξ acts as multiplicative identity since $A \cap \xi = A$ and ϕ acts as additive identity (i.e. zero element) since $A \cup \phi = A, \forall A \subseteq \xi$. Also note that in boolean algebra, inverse of a is generally denoted as a' and not a^{-1} .
3. The distributive property $a \cdot (b + c) = a \cdot b + a \cdot c$ is similar to that of real numbers, but the distribution of $+$ over $\cdot$ i.e. $a + (b \cdot c) = (a + b) \cdot (a + c)$ does not hold for conventional algebra.
4. In conventional algebra, $x + x + x + \dots$ n times $= n x$, but in boolean algebra, $x + x + x + \dots$ n times $= x$. This is called idempotent law.
5. Since family of sets is a classical example of a boolean algebra, many texts use $\cup$ and $\cap$ instead of $+$ and $\cdot$.

6. Modern computers and telecommunications use boolean algebra a lot — binary digits (or bits) 0 and 1 correspond to electrical switch off or on, current absent or flowing, bulbs off or on, capacitor discharged or charged etc. This will become clearer as we proceed.

7. Sometimes $a \cdot b$ is written simply as ab .

8. When parentheses are not used, the operation $\cdot$ has precedence over $+$.

Thus, in $a + b \cdot c$, we first evaluate $b \cdot c$.

Example. Let $B = \{0, 1\}$ and let two operations $+$ and $\cdot$ be defined on B as follows :

$+$	1	0
1	1	1
0	1	0

$\cdot$	1	0
1	1	0
0	0	0

Then B , or more precisely the triplet $(B, +, \cdot)$ is a boolean algebra. (Check ! Are all axioms (1) to (6) satisfied?) Here 1 is the multiplicative identity and 0 is the additive identity. This is the smallest possible Boolean Algebra called Switching Algebra.

More examples of Boolean Algebras

We have already given one example of a Boolean Algebra $B = \{0, 1\}$.

Now as second example, let A be a family of sets which is closed under the operations of union, intersection and complement. Then $(A, \cup, \cap)$ is a Boolean Algebra. Note that universal set ξ is the unit element and the null set ϕ is the zero element.

ILLUSTRATIVE EXAMPLES

Example 1. If $V = \{1, 2, 3\}$, $A = \{1, 2\}$, then $A' = \{3\}$. Show that the set $T = \{V, A, A', \phi\}$ along with operations $\cup$ and $\cap$ forms Boolean algebra.

Solution. We have to show that this system satisfies basic axioms of Boolean Algebra.

The composition tables for $\cup$ and $\cap$ are as follows :

$\cup$	V	A	A'	ϕ
V	V	V	V	V
A	V	A	V	A
A'	V	V	A'	A'
ϕ	V	A	A'	ϕ

$\cap$	V	A	A'	ϕ
V	V	A	A'	ϕ
A	A	A	ϕ	ϕ
A'	A'	ϕ	A'	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ

(1) Closure : all unions and intersections of sets V, A, A', ϕ belong to T .

(2) Operations $\cup$ and $\cap$ are commutative in sets.

(3) Operations $\cup$ and $\cap$ are associative in sets.

(4) Operations $\cup$ and $\cap$ are distributive in sets

(i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ etc.)

(5) Identity element for $\cup$, that is, zero element (or additive identity) is ϕ , since $V \cup \phi = V$; $A \cup \phi = A$; $A' \cup \phi = A'$; $\phi \cup \phi = \phi$. Identity element for operation $\cap$ (that is, unit element) is V since $V \cap V = V$; $A \cap V = A$; $A' \cap V = A'$; $\phi \cap V = \phi$.

(6) Inverse : Inverse of A is A' and inverse of V is ϕ (check).

Since all basic axioms are satisfied, the system $(T, \cup, \cap)$ is a boolean algebra.

Example 2. Let D_n denote the set of divisors of n , where n is a natural number, define operations $+$, $\cdot$ and $'$ on D_n as

$a + b = \text{lcm } \{a, b\}$ i.e. least common multiple of a and b ,

$a \cdot b = \text{gcd } \{a, b\}$ i.e. greatest common divisor of a and b , and $a' = \frac{n}{a}$.

Prove that D_4 is not a boolean algebra, while D_6 is a boolean algebra.

Solution. (i) For $D_4 = \{1, 2, 4\}$, the tables for given operations are

+	1	2	4
1	1	2	4
2	2	2	4
4	4	4	4

$\cdot$	1	2	4
1	1	1	1
2	1	2	2
4	1	2	4

$'$	1	2	4
	4	2	1

From these we observe that operations $+$ and $\cdot$ are closed, commutative and associative. For distribution of $+$ over $\cdot$, we see that

$$1 + (2 \cdot 4) = 1 + 2 = 2;$$

$$(1 + 2) \cdot (1 + 4) = 2 \cdot 4 = 2 \text{ etc.}$$

Similarly, we can verify for other elements, and also the distribution of $\cdot$ over $+$.

Now zero element (additive identity) is such that

$$a + \text{zero element} = a \quad \forall a \in D_4.$$

From table of $+$, we see that $a + 1 = a \quad \forall a$, so that 1 acts as zero element (additive identity), while from table of $\cdot$ we see that $a \cdot 4 = a \quad \forall a$, so that 4 acts as unit element (multiplicative identity).

Thus, we should have $a + a' = \text{unit element}$, $a \cdot a' = \text{zero element} \quad \forall a$.

From table of $'$, we see that $2' = 2$.

However from table of $+$, we see that $2 + 2 = 2 \neq 4$ (unit element).

Also from table of $\cdot$, we see that $2 \cdot 2 = 2 \neq 1$ (zero element).

Hence, D_4 is not a boolean algebra.

(ii) For $D_6 = \{1, 2, 3, 6\}$, we construct tables for operations as

+	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

$\cdot$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

$'$	1	2	3	6
	6	3	2	1

As above, we can verify closure, commutative, associative and distributive properties.

From $+$ table, we see that $a + 1 = a \quad \forall a$, so that 1 acts as zero element (additive identity), while from $\cdot$ table, we see that $a \cdot 6 = a \quad \forall a$, so that 6 acts as unit element (multiplicative identity).

Now we verify $a + a' = \text{unit element}$, $a \cdot a' = \text{zero element} \quad \forall a$:

$$1 + 6 = 6, 1 \cdot 6 = 1$$

$$2 + 3 = 6, 2 \cdot 3 = 1$$

$$3 + 2 = 6, 3 \cdot 2 = 1$$

$$6 + 1 = 6, 6 \cdot 1 = 1$$

Thus, all properties of boolean algebra are satisfied.

Note. In general, if n is a product of distinct prime numbers, then $D_n = \{\text{set of positive divisors of } n\}$ is a boolean algebra, with least common multiple acting as $+$, greatest common divisor acting as $\cdot$, $\frac{n}{a}$ acting as complement of a , integer 1 acting as zero element (additive identity) and integer n acting as unit element (multiplicative identity). However, if in the prime factorisation of n , any number is repeated, then D_n is not a boolean algebra.

EXERCISE 3.1

1. If $B = \{\phi, \xi, S, S'\}$ where S is any non-empty subset of ξ , then show that B along with the operations $\cup$ and $\cap$ forms a boolean algebra.
2. Let $B = \{\{1\}, \{2\}, \{1, 2\}, \phi\}$. Show that $(B, \cap, \cup, ', \phi, \{1, 2\})$ is a boolean algebra.
3. Let $B = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Show that $(B, \cup, \cap, ', \phi, \{1, 2, 3\})$ is a boolean algebra.
4. Let $D_n =$ set of divisors of n , where n is a natural number.

Define operations $+$, $\cdot$ and $'$ as

$$a + b = \text{lcm } \{a, b\}, a \cdot b = \text{gcd } \{a, b\}, a' = \frac{n}{a}.$$

Prove that D_8 is not a boolean algebra while D_{15} is a boolean algebra.

3.2 DUALITY IN A BOOLEAN ALGEBRA

By definition, the **dual** of any statement in a boolean algebra is the statement derived by interchanging $+$ and $\cdot$ and also interchanging the identities 1 and 0 in the original statement.

For example, the dual of statement $a + b = b + a$ is $a \cdot b = b \cdot a$, and dual of statement $a + a = a$ is $a \cdot a = a$.

The **Principle of Duality** says that the *dual of any theorem in a Boolean algebra is also a theorem.*

A theorem is derived from the basic axioms; dual statement of theorem can be proved by using the dual of each step of the proof of original theorem. For example, if we can prove that $a + a = a$, it will follow that $a \cdot a = a$; if we can prove that $a + 1 = 1$, it will follow that $a \cdot 0 = 0$. Thus, when we prove a theorem in boolean algebra, we get its dual for free.

3.3 ELEMENTARY PROPERTIES OF BOOLEAN ALGEBRA

Starting from basic axioms (1) to (6), a number of properties of Boolean Algebra can be proved. Important of these are :

- (i) **(Idempotent Law)** : $a + a = a$ and $a \cdot a = a$.
- (ii) **(Boundedness Laws)** $a + 1 = 1$ and $a \cdot 0 = 0$.
- (iii) **(Involution Law)** : $(a')' = a$.
- (iv) $1' = 0$ and $0' = 1$.
- (v) **(De Morgan's Laws)** : $(a + b)' = a' \cdot b'$ and $(a \cdot b)' = a' + b'$.
- (vi) **(Law of absorption)** : $a + (a \cdot b) = a$ and $a \cdot (a + b) = a$.
- (vii) **(Uniqueness of inverse)** : for every $a \in B$, a' is unique.

Proofs of properties:

- (i) To prove that $a + a = a$

$$\begin{aligned} a &= a + 0 && \text{(by Identity)} \\ &= a + (a \cdot a') && \text{(by Inverse)} \\ &= (a + a) \cdot (a + a') && \text{(by Distributive Law)} \\ &= (a + a) \cdot 1 && \text{(by Inverse)} \\ &= a + a && \text{(by Identity)} \end{aligned}$$

To prove that $a \cdot a = a$

$$\begin{aligned} a &= a \cdot 1 && \text{(by Identity)} \\ &= a \cdot (a + a') && \text{(by Inverse)} \\ &= a \cdot a + a \cdot a' && \text{(by Distributive Law)} \\ &= a \cdot a + 0 && \text{(by Inverse)} \\ &= a \cdot a && \text{(by Identity)} \end{aligned}$$

Aliter, the property $a \cdot a = a$ follows by duality from $a + a = a$.

(ii) To prove that $a + 1 = 1$

$$\begin{aligned}
 1 &= a + a' && \text{(by Inverse)} \\
 &= a + a' \cdot 1 && \text{(by Identity)} \\
 &= (a + a') \cdot (a + 1) && \text{(by Distributive Law)} \\
 &= 1 \cdot (a + 1) && \text{(by Inverse)} \\
 &= a + 1 && \text{(by Identity)}
 \end{aligned}$$

The property $a \cdot 0 = 0$ follows by duality.

Exercise : Prove $a \cdot 0 = 0$ from basic axioms!

(iii) To prove Involution Law : $(a')' = a$

As a' is inverse (complement) of a , we have

$$a + a' = 1 \text{ and } a \cdot a' = 0.$$

It can be written as $a' + a = 1$ and $a' \cdot a = 0$.

Therefore, by axiom (6), $(a')' = a$.

(iv) To prove that $1' = 0$ and $0' = 1$

$$\begin{aligned}
 1' &= 1' \cdot 1 && \text{(by Identity)} \\
 &= 1 \cdot 1' && \text{(by Commutative Law)} \\
 &= 0 && \text{(by Inverse)}
 \end{aligned}$$

The property $0' = 1$ follows by duality.

Exercise : Prove $0' = 1$ from basic axioms!

Remark. To prove that $a' = b$ or $b' = a$ i.e. to prove that a and b are inverse of each other, we should prove $a + b = 1$ and $a \cdot b = 0$.

(v) De Morgan's Laws : To prove that $(a + b)' = a' \cdot b'$ (I.S.C. 2000)

$$\begin{aligned}
 (a + b) \cdot (a' \cdot b') &= (a' \cdot b') \cdot (a + b) && \text{(by Commutative Law)} \\
 &= ((a' \cdot b') \cdot a) + ((a' \cdot b') \cdot b) && \text{(by Distributive Law)} \\
 &= ((b' \cdot a') \cdot a) + ((a' \cdot b') \cdot b) && \text{(why?)} \\
 &= (b' \cdot (a' \cdot a)) + (a' \cdot (b' \cdot b)) && \text{(why?)} \\
 &= (b' \cdot 0) + (a' \cdot 0) = 0 + 0 = 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{Also } (a + b) + (a' \cdot b') &= (a + b + a') \cdot (a + b + b') \\
 &= (a + a' + b) \cdot (a + 1) = (1 + b) \cdot 1 = 1 \cdot 1 = 1.
 \end{aligned}$$

Hence by definition of inverse, $(a + b)' = a' \cdot b'$.

To prove that $(a \cdot b)' = a' + b'$

$$\begin{aligned}
 (a \cdot b) + (a' + b') &= a' + b' + a \cdot b && \text{(by Commutative Law)} \\
 &= (a' + b' + a) \cdot (a' + b' + b) && \text{(why?)} \\
 &= (a + a' + b') \cdot (a' + b + b') && \text{(why?)} \\
 &= (1 + b') \cdot (a' + 1) && \text{(why?)} \\
 &= 1 \cdot 1 = 1.
 \end{aligned}$$

$$\begin{aligned}
 \text{Also } (a \cdot b) \cdot (a' + b') &= (a \cdot b) \cdot a' + (a \cdot b) \cdot b' \\
 &= (a \cdot a') \cdot b + a \cdot (b \cdot b') \\
 &= 0 \cdot b + a \cdot 0 = 0 + 0 = 0.
 \end{aligned}$$

Hence by definition of inverse, $(a \cdot b)' = a' + b'$.

Aliter, the property $(a \cdot b)' = a' + b'$ follows by duality from $(a + b)' = a' \cdot b'$.

(vi) Law of Absorption : To prove that $a + a \cdot b = a$

$$\begin{aligned}
 a + (a \cdot b) &= a \cdot 1 + a \cdot b && \text{(why?)} \\
 &= a \cdot (1 + b) && \text{(by Distributive Law)} \\
 &= a \cdot 1 && \text{(by property (ii) above)} \\
 &= a && \text{(why?)}
 \end{aligned}$$

To prove that $a \cdot (a + b) = a$

$$\begin{aligned} a \cdot (a + b) &= (a + 0) \cdot (a + b) && \text{(why?)} \\ &= a + 0 \cdot b && \text{(by Distributive Law)} \\ &= a + 0 && \text{(why?)} \\ &= a && \text{(why?)} \end{aligned}$$

Aliter, the property $a \cdot (a + b) = a$ follows by duality from $a + a \cdot b = a$.

(vii) Uniqueness of complement (inverse) : (I.S.C. 2003)

Let us assume that a'_1 and a'_2 are two complements of a . Then by definition,
 $a + a'_1 = 1$, $a \cdot a'_1 = 0$, $a + a'_2 = 1$, $a \cdot a'_2 = 0$.

$$\begin{aligned} a'_1 &= a'_1 \cdot 1 = a'_1 \cdot (a + a'_2) \\ &= (a'_1 \cdot a) + (a'_1 \cdot a'_2) = 0 + (a'_1 \cdot a'_2) \\ &= (a \cdot a'_2) + (a'_1 \cdot a'_2) = (a + a'_1) \cdot a'_2 \\ &= 1 \cdot a'_2 = a'_2. \end{aligned}$$

Now try to prove it by starting from $a'_1 = a'_1 + 0$.

ILLUSTRATIVE EXAMPLES

Example 1. Write the dual of each of the following statements :

- (i) $x + (y \cdot x) = x$
- (ii) $x \cdot y' + y = x + y$
- (iii) $(x + y) \cdot (x + 1) = x + x \cdot y + y$
- (iv) $(x' + y) \cdot (x + y') = x' \cdot y' + x \cdot y$
- (v) $[(x' + y) \cdot (y' + z)] \cdot (x' + z)' = 0$.

Solution. Since the dual statement is derived by interchanging $+$ and $\cdot$ and also interchanging the identities 1 and 0 in the original statement, therefore, the dual statements of the given statements are :

- (i) $x \cdot (y + x) = x$
- (ii) $(x + y') \cdot y = x \cdot y$
- (iii) $(x \cdot y) + (x \cdot 0) = x \cdot (x + y) \cdot y$
- (iv) $(x' \cdot y) + (x \cdot y') = (x' + y') \cdot (x + y)$
- (v) $[(x' \cdot y) + (y' \cdot z)] + (x' \cdot z)' = 1$.

Example 2. Prove that $a + (a' \cdot b) = a + b$ and $a \cdot (a' + b) = a \cdot b$.

$$\begin{aligned} \text{Solution. } a + (a' \cdot b) &= (a + a') \cdot (a + b) && \text{(by Distributive Law)} \\ &= 1 \cdot (a + b) && \text{(by Complement)} \\ &= a + b && \text{(by Identity)} \end{aligned}$$

$$\begin{aligned} \text{Similarly, } a \cdot (a' + b) &= (a \cdot a') + (a \cdot b) && \text{(by Distributive Law)} \\ &= 0 + (a \cdot b) && \text{(by Complement)} \\ &= a \cdot b && \text{(by Identity)} \end{aligned}$$

Example 3. Prove that $a + (a' \cdot c + b) = a + b + c$.

$$\begin{aligned} \text{Solution. } a + (a' \cdot c + b) &= (a + a' \cdot c) + b && \text{(by Associative Law)} \\ &= ((a + a') \cdot (a + c)) + b && \text{(by Distributive Law)} \\ &= (1 \cdot (a + c)) + b && \text{(by Complement)} \\ &= (a + c) + b && \text{(by Identity)} \\ &= a + (c + b) && \text{(by Associativity)} \\ &= a + (b + c) && \text{(by Commutative Law)} \\ &= a + b + c. \end{aligned}$$

Example 4. Prove that $ab + c(a' + b') = ab + c$.

(I.S.C. 2004)

Solution. $ab + c(a' + b') = ab + c(ab)'$

(by De Morgan's Law)

$$= (ab + c) \cdot (ab + (ab)')$$

(by Distributive Law)

$$= (ab + c) \cdot 1$$

(by Complement)

$$= ab + c$$

(by Identity)

Example 5. Prove that $(x + y) \cdot (x + 1) = x + x \cdot y + y$.

Solution. $(x + y) \cdot (x + 1) = (x + y) \cdot 1$

(by Boundedness)

$$= x + y$$

(by Identity)

$$= (x + x \cdot y) + y$$

(by Absorption Law)

$$= x + x \cdot y + y$$

(by Associative Law)

Example 6. Show that in a boolean algebra, the zero element 0 and the unit element 1 are unique.

Solution. Let 0, $\bar{0}$ be two zero elements in B.

As 0 is an additive identity, by definition,

$$a + 0 = a \text{ for all } a \in B.$$

$$\therefore \bar{0} + 0 = \bar{0}$$

...(i) (taking $a = \bar{0}$)

Also as $\bar{0}$ is an additive identity, by definition,

$$a + \bar{0} = a \text{ for all } a \in B.$$

$$\therefore 0 + \bar{0} = 0$$

...(ii) (taking $a = 0$)

However, by commutativity, we have $\bar{0} + 0 = 0 + \bar{0}$

$$\Rightarrow \bar{0} = 0.$$

(using (i) and (ii))

Hence, zero element in a boolean algebra is unique.

Let 1, $\bar{1}$ be two unit elements in B. Then, by definition

$$a \cdot 1 = a \text{ and } a \cdot \bar{1} = a \text{ for all } a \in B.$$

$$\therefore \bar{1} \cdot 1 = \bar{1} \text{ and } 1 \cdot \bar{1} = 1.$$

However, by commutativity, we have $\bar{1} \cdot 1 = 1 \cdot \bar{1}$.

$$\text{Hence } \bar{1} = 1.$$

Example 7. Prove the following :

(i) If $x + y = 0$, then $x = 0 = y$

(ii) $x \cdot y' = 0$ if and only if $x \cdot y = x$

(iii) $x = 0$ if and only if $y = x \cdot y' + x' \cdot y$ for all y .

Solution. (i) Given $x + y = 0$

$$\text{Now } x = x \cdot (x + y)$$

(by Absorption Law)

$$= x \cdot 0$$

(given, $x + y = 0$)

$$= 0.$$

$$\text{Similarly, } y = y \cdot (y + x) = y \cdot (x + y) = y \cdot 0 = 0.$$

(ii) First assume $x \cdot y' = 0$

$$\Rightarrow (x \cdot y')' = 0' \Rightarrow x' + y = 1$$

$$\text{Now } x = x \cdot 1 = x \cdot (x' + y)$$

(using $x' + y = 1$)

$$= x \cdot x' + x \cdot y = 0 + x \cdot y$$

$$= x \cdot y.$$

Now to prove the other part, assume $x \cdot y = x$. We wish to prove that

$$x \cdot y' = 0$$

$$\text{Now } x \cdot y' = (x \cdot y) \cdot y'$$

(using $x = x \cdot y$)

$$= x \cdot (y \cdot y') = x \cdot 0 = 0.$$

(iii) First assume that $x = 0$

Then $x \cdot y' + x' \cdot y = 0 \cdot y' + 0' \cdot y = 0 + 1 \cdot y = 0 + y = y$ for all y .

To prove the other part, assume that $y = x \cdot y' + x' \cdot y$ for all y .

In particular, it is true for $y = 0$.

$$\therefore 0 = x \cdot 0' + x' \cdot 0 = x \cdot 1 + x' \cdot 0 = x + 0 = x.$$

Thus, $x = 0$.

Example 8. Prove that if $x + y = x + z$ and $x' + y = x' + z$, then $y = z$.

Solution. Given $x + y = x + z$

...(i)

and $x' + y = x' + z$

...(ii)

Now $y = y + 0 = y + x \cdot x'$

($\because x \cdot x' = 0$)

$$= (y + x) \cdot (y + x')$$

(by Distributive Law)

$$= (x + y) \cdot (x' + y)$$

(by Commutative Law)

$$= (x + z) \cdot (x' + z)$$

(by using (i) and (ii))

$$= x \cdot x' + z$$

(by Distributive Law)

$$= 0 + z = z.$$

Example 9. A boolean algebra cannot have exactly three elements.

Solution. Let a boolean algebra B have exactly three elements and let $B = \{0, 1, x\}$, where $x \neq 0, x \neq 1$ be a boolean algebra under the operation $+$ and $\cdot$.

Since $x \in B$, there exists $x' \in B$ such that

$$x + x' = 1 \quad \dots(i) \quad \text{and} \quad x \cdot x' = 0$$

...(ii)

As $x' \in B$ and B has exactly three elements, namely 0, 1 and x .

Three cases arise :

Case I. If $x' = 0$

then $(x')' = 0' \Rightarrow x = 1$, which is wrong.

Case II. If $x' = 1$

then $(x')' = 1' \Rightarrow x = 0$, which is wrong.

Case III. If $x' = x$, then using (i), we get

$$x' + x = 1 \Rightarrow x + x = 1$$

$$\Rightarrow x = 1$$

($\because x + x = x$, Idempotent law)

which is wrong.

It follows that a boolean algebra cannot have exactly three elements.

3.4 BOOLEAN EXPRESSIONS AND FUNCTIONS

In a boolean algebra $(B, +, \cdot, ', 0, 1)$, the specific elements like 0 and 1, or other elements of B, are called **constants**. **Variables** like x and y may denote any element of B. Thus in boolean algebra $\{0, 1\}$, there are only two constants 0, 1, while we may define any number of variables like x, y, z etc. on B, which may take values 0 or 1.

A **boolean expression** is any expression built from variables and constants by applying the operations $+$, $\cdot$, $'$ a finite number of times. Examples of boolean expressions are

$$x + 1, x + y, x' + y \cdot 0, (x + y) (y' + z), [(x_1 \cdot x_2) + x_3]' \text{ etc.}$$

A **boolean function** is determined by a boolean expression.

For example,

$$f(x) = x'$$

$$f(x, y) = x + y$$

$$g(x, y) = x \cdot y' + 1$$

Two different boolean expressions may represent the same boolean function. For example, $x \cdot (y + z)$ and $x \cdot y + x \cdot z$ represent the same boolean function in 3 variables.

A boolean function of n variables $x_1, x_2, \dots, x_n$ is a mapping from B^n to B. We write it as $f(x_1, x_2, \dots, x_n)$ or $g(x_1, x_2, \dots, x_n)$ etc.

ILLUSTRATIVE EXAMPLES

Example 1. Find the value of the boolean expression $(x_1 \cdot x_2)' + x_3$ if

(i) $x_1 = 1, x_2 = 0, x_3 = 1$

(ii) $x_1 = 1, x_2 = 1, x_3 = 0.$

Solution. (i) $(x_1 \cdot x_2)' + x_3 = (1 \cdot 0)' + 1 = 0' + 1$
 $= 1 + 1 = 1.$

(ii) $(x_1 \cdot x_2)' + x_3 = (1 \cdot 1)' + 0 = 1' + 0$
 $= 0 + 0 = 0.$

Example 2. Construct input/output table for the boolean function f on boolean algebra $\{0, 1\}$ defined by :

(i) $f(x_1, x_2) = x_1 \cdot x_2'$

(ii) $f(x_1, x_2, x_3) = (x_1 \cdot x_2)' + x_3.$

Solution. (i) As the given boolean function has two variables and is defined on the boolean algebra $\{0, 1\}$, its input/output table is :

Input			Output
x_1	x_2	x_2'	$x_1 \cdot x_2'$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

(ii) As the given boolean function has three variables and is defined on the boolean algebra $\{0, 1\}$, its input/output table is :

Input				Output	
x_1	x_2	x_3	$x_1 \cdot x_2$	$(x_1 \cdot x_2)'$	$(x_1 \cdot x_2)' + x_3$
1	1	1	1	0	1
1	1	0	1	0	0
1	0	1	0	1	1
1	0	0	0	1	1
0	1	1	0	1	1
0	1	0	0	1	1
0	0	1	0	1	1
0	0	0	0	1	1

Example 3. Define a boolean function f on boolean algebra $\{0, 1\}$ corresponding to the expression $x + y'$. Give its domain and range. Construct input/output table (or truth table).

Solution. We define $f : B^2 \rightarrow B$ as

$$f(x, y) = x + y', \text{ where } x, y \in \{0, 1\}$$

Domain is B^2 i.e. $(\{0, 1\})^2 = \{0, 1\} \times \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$

From $f(x, y) = x + y'$,

putting $x = 0, y = 0$, we get $f(0, 0) = 0 + 0' = 0 + 1 = 1.$

Similarly, $f(0, 1) = 0 + 1' = 0 + 0 = 0,$

$$f(1, 0) = 1 + 0' = 1 + 1 = 1,$$

$$f(1, 1) = 1 + 1' = 1 + 0 = 1.$$

$\therefore$ Range is B i.e. $\{0, 1\}.$

The input/output table or truth table is :

x	y	y'	$f(x, y) = x + y'$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

Example 4. (i) If $f(x, y, z) = xy + yz' + x'yz$ in boolean algebra $B = \{0, 1\}$, evaluate $f(0, 0, 0)$ and $f(0, 1, 1)$. How many rows would the truth table have?

(ii) Write a boolean expression in terms of x, y or their complements which has value 1 when $x = 0, y = 1$ and assumes value 0 otherwise.

Solution. (i) $f(x, y, z) = xy + yz' + x'yz$

$$\therefore f(0, 0, 0) = 0 \cdot 0 + 0 \cdot 0' + 0' \cdot 0 \cdot 0 = 0 + 0 \cdot 1 + 1 \cdot 0 \cdot 0 \\ = 0 + 0 + 0 = 0$$

$$f(0, 1, 1) = 0 \cdot 1 + 1 \cdot 1' + 0' \cdot 1 \cdot 1 = 0 + 1 \cdot 0 + 1 \cdot 1 \cdot 1 = 0 + 0 + 1 = 1.$$

As there are 3 variables, each of which can have two values, the truth table will have $2 \times 2 \times 2 = 8$ rows.

(ii) Here we use the fact that $x \cdot y$ has value 1 only when $x = 1, y = 1$; otherwise $x \cdot y$ has value 0. Hence, the expression which has value 1 when $x = 0, y = 1$ is $x'y$.

Note. This idea can be better understood by constructing the following truth table :

x	y	x'	y'	xy	xy'	$x'y$	$x'y'$
1	1	0	0	1	0	0	0
1	0	0	1	0	1	0	0
0	1	1	0	0	0	1	0
0	0	1	1	0	0	0	1

It is very clear from this table that expression $x'y$ has value 1 when $x = 0, y = 1$ and assumes value 0 otherwise.

EXERCISE 3.2

1. Write the dual statement of each of the following :

- (i) $(1 + x) \cdot (0 + y) = y$ (ii) $(x \cdot y)' = x' + y'$
 (iii) $a + 1 = 1$ (iv) $x + x'y = x + y$
 (v) $x + [x \cdot (y + 1)] = x$ (vi) $x + [(y' + x) \cdot y]' = 1$
 (vii) $(x + y)' = x' \cdot y'$ (viii) $(x' + y')' = x \cdot y$
 (ix) $(1 \cdot x) + 0 = x$ (x) $(1 \cdot x)' = 0 + x'$
 (xi) If $x + y = 0$ then $x = 0 = y$ (xii) $x \cdot y' = 0$ if and only if $x \cdot y = x$
 (xiii) $x = 0$ if and only if $y = x \cdot y' + x' \cdot y$ for all y .

2. Using elementary properties of boolean algebra, prove that :

- (i) $x + x \cdot (y + 1) = x$ (ii) $(a + b) a' b' = 0$
 (iii) $(x + y) + (x' \cdot y') = 1$.

3. Prove the following statements :

- (i) If $x \cdot y = 1$, then $x = 1 = y$ (ii) $x + y' = 1$ if and only if $x + y = x$.