11 Applications of Definite Integrals

Definite integrals have a wide range of applications. In this chapter, we shall use definite integrals in computing the *areas of bounded regions*.

11.1 AREAS OF BOUNDED REGIONS

If the function f is continuous and non-negative in the closed interval $[a, b]$, then the area of the region below the curve $y = f(x)$, above the x-axis and between the ordinates $x = a$ and $x = b$ or briefly the area of the region bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is given by $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.

Proof. Let AB be the curve $y = f(x)$ between AC$(x = a)$ and BD $(x = b)$, then the required area is the area of the shaded region ACDB.

Let P(x, y) be a point on the curve $y = f(x)$ and Q$(x + \delta x, y + \delta y)$ be a neighbouring point on the curve, then MP $= y$, NQ $= y + \delta y$ and MN $= \delta x$. Let A be the area of the region ACMP and A + δA be the area of the region ACNQ, then δA = area of region PMNQ.

Area of rectangle PMNR $= y\delta x$ and area of rectangle SMNQ $= (y + \delta y)\delta x$.

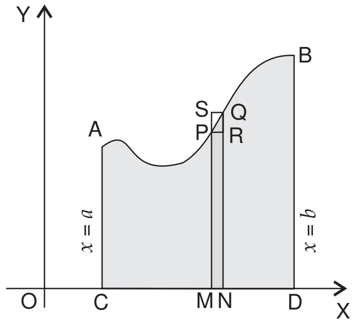

Fig. 11.1.

From fig. 11.1, area of rectangle PMNR ≤ area of region PMNQ ≤ area of rectangle SMNQ

$$\Rightarrow \quad y\delta x \le \delta A \le (y + \delta y)\delta x$$

$$\Rightarrow \quad y \le \frac{\delta A}{\delta x} \le y + \delta y \qquad \ldots(i)$$

When P → Q, $\delta x \to 0$, $\delta y \to 0$ and $\frac{\delta A}{\delta x} \to \frac{dA}{dx}$.

From (i), $\underset{\delta x \to 0}{\text{Lt}}\, y \le \underset{\delta x \to 0}{\text{Lt}} \frac{\delta A}{\delta x} \le \underset{\delta y \to 0}{\text{Lt}} (y + \delta y)$

$$\Rightarrow \quad y \le \frac{dA}{dx} \le y \Rightarrow y = \frac{dA}{dx}.$$

Integrating both sides w.r.t. x between the limits a to b, we get

$$\int_a^b y\,dx = \int_a^b \frac{dA}{dx}\,dx = [A]_a^b$$

$$= \text{(value of area A when } x = b) - \text{(value of area A when } x = a)$$

$$= \text{area ACDB} - 0 = \text{area ACDB}.$$

If a function f is continuous and non-positive in the closed interval $[a, b]$, then the curve $y = f(x)$ lies below the x-axis and the definite integral $\int_a^b f(x)\,dx$ is negative. Since the area of a region is always non-negative, the area of the region bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is given by $\left|\int_a^b f(x)\,dx\right|$ or $\left|\int_a^b y\,dx\right|$.

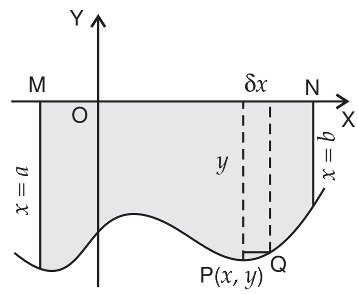

Fig. 11.2.

Hence, if the curve $y = f(x)$ is continuous and does not cross the x-axis, then the area of the region bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by $\left|\int_a^b f(x)\,dx\right|$ or $\left|\int_a^b y\,dx\right|$.

Similarly, if the curve $x = g(y)$ is continuous and does not cross the y-axis, then the area of the region bounded by the curve $x = g(y)$, the y-axis and the *abscissae* $y = c$, $y = d$ is given by

$$\left|\int_c^d g(y)\,dy\right| \text{ or } \left|\int_c^d x\,dy\right|.$$

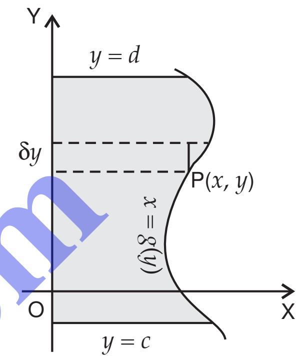

Fig. 11.3.

Remark. It may be noted that when sign of $f(x)$ is not known, then $\int_a^b f(x)\,dx$ may not represent the area *enclosed between* the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$, whereas $\int_a^b |f(x)|\,dx$ equals the area enclosed between the graph of the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$.

For example, let us consider the integrals $\int_{-1}^{1} x\,dx$ and $\int_{-1}^{1} |x|\,dx$.

First integral $= \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2}(1^2 - (-1)^2) = 0$, whereas second integral

$$= \int_{-1}^{1} |x|\,dx = \int_{-1}^{0} (-x)\,dx + \int_{0}^{1} x\,dx$$

(Common sense suggests this division as $|x| = -x$ in $[-1, 0]$ and $|x| = x$ in $[0, 1]$).

$$= \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = -\frac{1}{2}(0 - 1) + \frac{1}{2}(1 - 0) = 1.$$

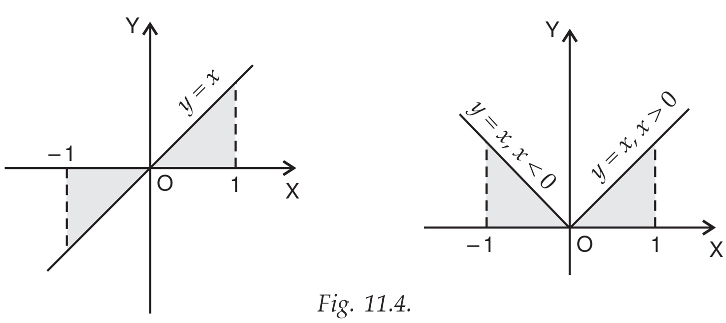

Fig. 11.4.

Clearly, the area enclosed between $y = x$, the x-axis and the ordinates $x = -1$ and $x = 1$ is not zero.

It follows that if the graph of a function f is continuous in $[a, b]$ and crosses the x-axis at finitely many points in $[a, b]$, then the area enclosed between the graph of the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is given by $\int_a^b |f(x)|\,dx$ or $\int_a^b |y|\,dx$.

11.1.1 Area bounded between curves

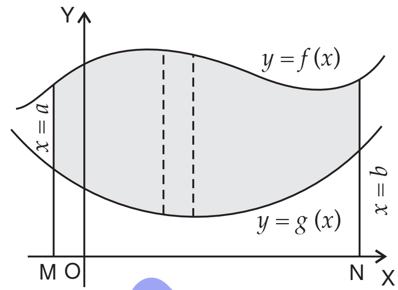

Fig. 11.5.

If $f(x)$, $g(x)$ are both continuous in $[a, b]$ and $0 \le g(x) \le f(x)$ for all $x \in [a, b]$, then the area of the region between the graphs of $y = f(x)$, $y = g(x)$ and the ordinates $x = a$, $x = b$ is given by

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b (f(x) - g(x))\,dx.$$

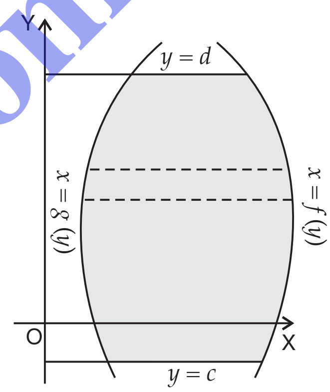

Fig. 11.6.

Similarly, the area of the region between the graphs of $x = f(y)$, $x = g(y)$ and the abscissae $y = c$, $y = d$ is given by $\int_c^d (f(y) - g(y))\,dy$.

Remarks

1. If $f(x)$, $g(x)$ are both continuous in $[a, b]$ and $g(x) \le f(x)$ for all $x \in [a, b]$, then the above formula also holds when one or both of the curves $y = f(x)$ and $y = g(x)$ lie partially or completely below the x-axis.
2. If the graphs of the curves $y = f(x)$ and $y = g(x)$ cross each other at finitely many points, then the area enclosed between the graphs of the two curves and the ordinates $x = a$ and $x = b$ is given by $\int_a^b |f(x) - g(x)|\,dx$.
3. Similarly, the area of the region between the graphs of $x = f(y)$, $x = g(y)$ and the abscissae $y = c$, $y = d$ is given by $\int_c^b |f(y) - g(y)|\,dy$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the area of the region bounded by $y^2 = 4x$, $x = 1$, $x = 4$ and the x-axis in the first quadrant.*

Solution. The given curve is $y^2 = 4x$ which represents a right hand parabola with vertex at (0, 0). The area bounded by $y^2 = 4x$, $x = 1$, $x = 4$ and the x-axis is shown shaded in the figure.

Required area $= \int_1^4 y\,dx = \int_1^4 2\sqrt{x}\,dx$

($\because y^2 = 4x \Rightarrow y = 2\sqrt{x}$ in the first quadrant)

$$= 2 \cdot \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_1^4 = \frac{4}{3}\left[4^{3/2} - 1^{3/2}\right] \text{ sq. unis}$$

$$= \frac{4}{3}[8-1] \text{ sq. units} = \frac{28}{3} \text{ sq. units.}$$

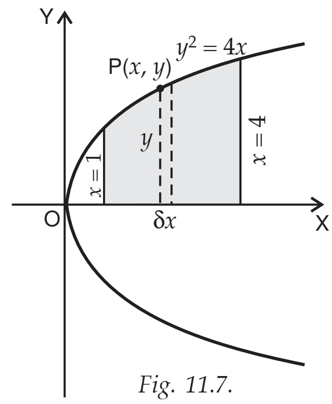

Fig. 11.7.

Example 2. *Draw a rough sketch of the curve $x^2 + y = 9$ and find the area enclosed by the curve, the x-axis and the lines $x + 1 = 0$ and $x - 2 = 0$.* *(I.S.C. 2009)*

Solution. The given curve is $x^2 + y = 9$ ...(*i*)

It can be written as $x^2 = 9 - y$

$\Rightarrow (x - 0)^2 = -(y - 9)$

which represents a downward parabola with vertex at (0, 9).

The parabola meets the *x*-axis *i.e.* $y = 0$ at $x^2 = 9$ *i.e.* at $x = -3, 3$.

A rough sketch of the curve is shown in fig. 11.8.

The given lines are $x + 1 = 0$ and $x - 2 = 0$ *i.e.* $x = -1$ and $x = 2$.

The area enclosed by the curve, the *x*-axis and the given lines is shown shaded in fig. 11.8.

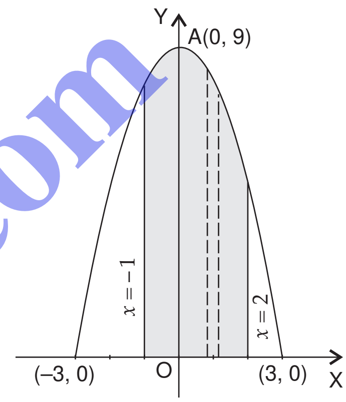

Fig. 11.8.

$\therefore$ Required area $= \int_{-1}^{2} y\,dx = \int_{-1}^{2} (9 - x^2)\,dx$ (using (*i*))

$$= \left[9x - \frac{x^3}{3}\right]_{-1}^{2} = \left(\left(18 - \frac{8}{3}\right) - \left(-9 + \frac{1}{3}\right)\right) \text{ sq. units}$$

$$= \left(27 - \frac{8}{3} - \frac{1}{3}\right) \text{ sq. units} = 24 \text{ sq. units.}$$

Example 3. *Determine the area enclosed between the curve $y = 4x - x^2$ and the x-axis.*

Solution. Given curve is $y = 4x - x^2$.

It can be written as $x^2 - 4x = -y \Rightarrow (x - 2)^2 = -(y - 4)$

which represents a downward parabola with vertex at (2, 4).

The parabola meets *x*-axis *i.e.* $y = 0$ at $4x - x^2 = 0$ *i.e.* at $x = 0, x = 4$.

$\therefore$ The area enclosed between the curve and the *x*-axis

$$= \int_0^4 y\,dx = \int_0^4 (4x - x^2)\,dx = \left[4 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_0^4$$

$$= \left(32 - \frac{64}{3}\right) - (0 - 0) = \frac{32}{3} \text{ sq. units.}$$

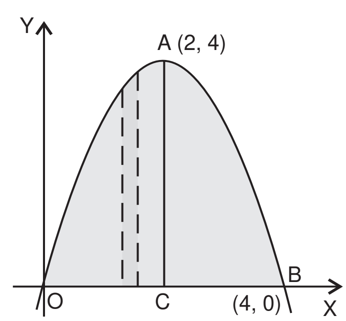

Fig. 11.9.

Alternatively. Since the parabola is symmetrical about the line $x = 2$,

$$\text{required area} = 2\int_0^2 (4x - x^2)\,dx = 2\left[4.\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= 2\left[\left(8-\frac{8}{3}\right)-(0-0)\right] \text{ sq. units } = 2.\frac{16}{3} \text{ sq. units } = \frac{32}{3} \text{ sq. units.}$$

Remark. *In case of symmetrical closed area, find the area of the smallest part and multiply the result by the number of symmetrical parts.*

Example 4. *Draw a rough sketch of the curve $y^2 + 1 = x$, $x \le 2$. Find the area enclosed by the curve and the line $x = 2$.* *(I.S.C. 2008)*

Solution. Given curve is $y^2 + 1 = x$.

It can be written as $y^2 = x - 1$, which represents a right hand parabola with vertex at A(1, 0).

The parabola meets the line $x = 2$ at when $y^2 = 1$ *i.e.* $y = 1, -1$.

A rough sketch of the curve $y^2 + 1 = x, x \le 2$ is shown in fig. 11.10. The area bounded by the curve $y^2 = x - 1$ and the line $x = 2$ is shown shaded in the figure. Since the given area is symmetrical about x-axis,

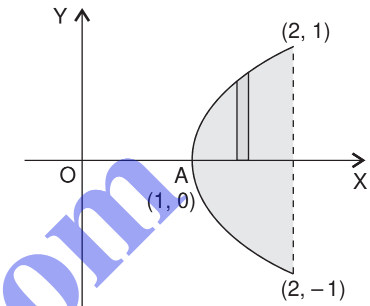

Fig. 11.10.

required area = 2(area of the region bounded by the curve $y^2 = x - 1$, the x-axis and the line $x = 2$)

$$= 2\int_1^2 y\,dx = 2\int_1^2 \sqrt{x-1}\,dx \qquad (\because y^2 = x - 1 \Rightarrow y = \sqrt{x-1} \text{ in the first quadrant})$$

$$= 2\,.\left[\frac{(x-1)^{3/2}}{\frac{3}{2}}\right]_1^2 = \frac{4}{3}[1^{3/2} - 0] \text{ sq. units } = \frac{4}{3} \text{ sq. units.}$$

Example 5. *Draw a rough sketch of the curve $y = x^2 - 5x + 6$ and find the area bounded by the curve and the x-axis.* *(I.S.C. 2010)*

Solution. The given curve is $y = x^2 - 5x + 6$.

It can be written as $x^2 - 5x + \frac{25}{4} = y + \frac{1}{4}$

$\Rightarrow \quad \left(x - \frac{5}{2}\right)^2 = y - \left(-\frac{1}{4}\right)$, which represents an upward parabola with vertex at $\left(\frac{5}{2}, -\frac{1}{4}\right)$.

A rought sketch of the curve is shown in fig. 11.11.

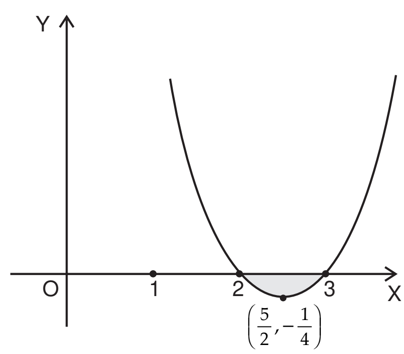

Fig. 11.11.

The parabola meets the x-axis *i.e.* $y = 0$ at

$x^2 - 5x + 6 = 0$ *i.e.* at $(x - 2)(x - 3) = 0$

i.e. at $x = 2$, $x = 3$.

As the required portion of the curve lies below x-axis, y is negative.

$$\therefore \quad \text{Required area} = \left|\int_2^3 y\,dx\right| = \left|\int_2^3 (x^2 - 5x + 6)dx\right|$$

$$= \left|\left[\frac{x^3}{3} - 5.\frac{x^2}{2} + 6x\right]_2^3\right| = \left|\left(9 - \frac{45}{2} + 18\right) - \left(\frac{8}{3} - 10 + 12\right)\right| \text{ sq. units}$$

$$= \left|\frac{9}{2} - \frac{14}{3}\right| \text{ sq. units } = \left|-\frac{1}{6}\right| \text{ sq. units } = \frac{1}{6} \text{ sq. units.}$$

Example 6. *Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.*

Solution. The given curve is $y^2 = 4x$ which represents a right hand parabola with vertex (0, 0). The area bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is shown shaded in fig. 11.12.

$$\text{Required area} = \int_0^3 x\, dy = \int_0^3 \frac{y^2}{4}\, dy$$

$$\left(\because\ y^2 = 4x \Rightarrow x = \frac{y^2}{4}\right)$$

$$= \frac{1}{4} \cdot \left[\frac{y^3}{3}\right]_0^3 = \frac{1}{12}\,[27 - 0]$$

$$= \frac{9}{4} \text{ sq. units.}$$

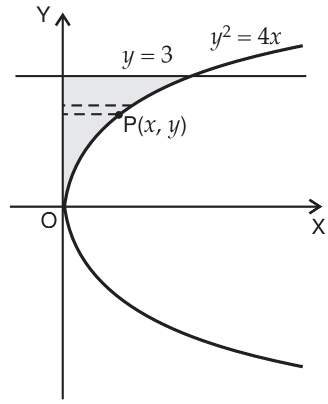

Fig. 11.12.

Example 7. *Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.*

Solution. The given curve is $y = x^2$ which represents an upward parabola with vertex at (0, 0). The area bounded by the curve and the line $y = 4$ is shown shaded in fig. 11.13.

Since the area is symmetrical about y-axis,

required area = 2 (area of the region bounded by $y = x^2$, the y-axis and the line $y = 4$)

$$= 2\int_0^4 x\, dy = 2\int_0^4 \sqrt{y}\, dx$$

$$(\because\ x^2 = y \Rightarrow x = \sqrt{y} \text{ in the first quadrant})$$

$$= 2 \cdot \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 = \frac{4}{3}\left[4^{3/2} - 0\right] = \frac{4}{3}\,[8 - 0] = \frac{32}{3} \text{ sq. units.}$$

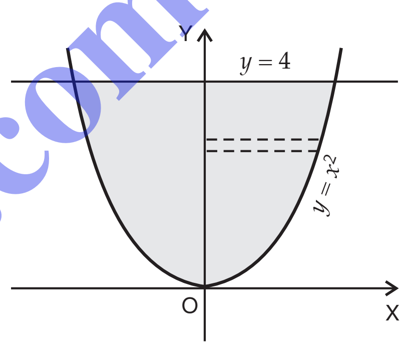

Fig. 11.13.

Example 8. *Sketch and shade the area of the region lying in the first quadrant and bounded by $y = 9x^2$, $x = 0$, $y = 1$ and $y = 4$. Find the area of the shaded region.* (I.S.C. 2004)

Solution. The given curve is $y = 9x^2$. It can be written as $x^2 = \frac{y}{9}$ which represents an upward parabola with vertex at (0, 0). The area lying in the first quadrant and bounded by $y = 9x^2$, $x = 0$, $y = 1$ and $y = 4$ is shown shaded in fig. 11.14.

$$\text{The required area} = \int_1^4 x\, dy = \int_1^4 \sqrt{\frac{y}{9}}\, dy$$

$$\left(\because\ x^2 = \frac{y}{9} \Rightarrow x = \sqrt{\frac{y}{9}} \text{ in the first quadrant.}\right)$$

$$= \frac{1}{3}\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_1^4 = \frac{2}{9}\left[4^{3/2} - 1^{3/2}\right]$$

$$= \frac{2}{9}\,(8 - 1) = \frac{14}{9} \text{ sq. units.}$$

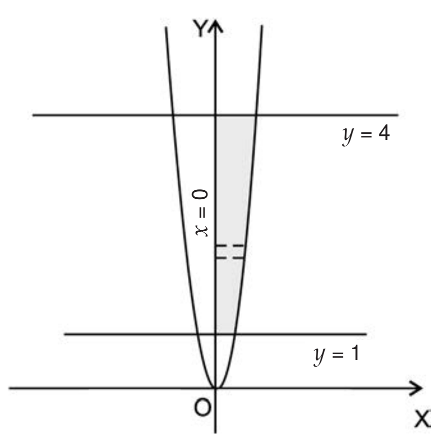

Fig. 11.14.

Example 9. *Find the area bounded by the curve $x = 8 + 2y - y^2$, the y-axis and the lines $y = -1$, $y = 3$.*

Solution. The given curve is $x = 8 + 2y - y^2$.

It can be written as

$$y^2 - 2y = -x + 8$$

$\Rightarrow (y - 1)^2 = -(x - 9)$ which represents a left hand parabola with vertex at (9, 1).

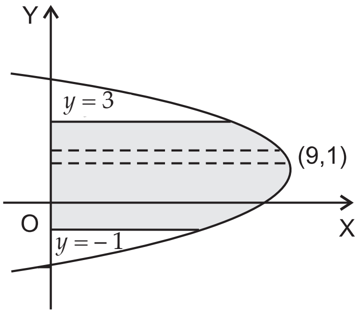

Fig. 11.15.

Required area

$$= \int_{-1}^{3} x\, dy = \int_{-1}^{3} (8 + 2y - y^2)\, dy$$

$$= \left[8y + 2.\frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^{3}$$

$$= (24 + 9 - 9) - \left(-8 + 1 + \frac{1}{3}\right) = \frac{92}{3} \text{ sq. units.}$$

Example 10. *Draw a rough sketch of the graph of the function $y = 2\sqrt{1 - x^2}$, $x \in [0, 1]$ and evaluate the area enclosed between the curve and the axes.*

Solution. The given curve is $y = 2\sqrt{1 - x^2}$

$\Rightarrow \frac{y^2}{4} = 1 - x^2 \Rightarrow \frac{x^2}{1} + \frac{y^2}{4} = 1$, which represents an ellipse of the second standard form. Hence, the given equation $y = 2\sqrt{1 - x^2}$ represents the portion of the ellipse lying in the first quadrant. Its rough sketch is shown in fig. 11.16.

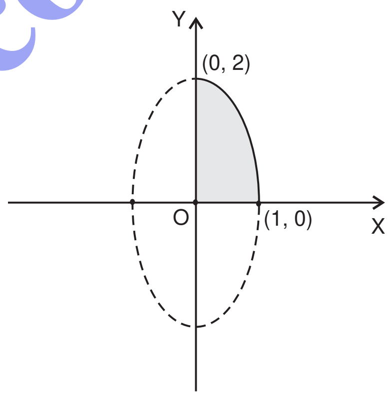

Fig. 11.16.

The required area = the area of the shaded region

$$= \int_{0}^{1} y\, dx = \int_{0}^{1} 2\sqrt{1 - x^2}\, dx$$

$$= 2\left[\frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = \left[x\sqrt{1 - x^2} + \sin^{-1} x\right]_0^1$$

$$= (0 + \sin^{-1} 1) - (0 + \sin^{-1} 0) = \frac{\pi}{2} \text{ sq. units.}$$

Example 11. *Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and $x = ae$ where $b^2 = a^2(1 - e^2)$ and $0 < e < 1$.*

Solution. The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \quad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

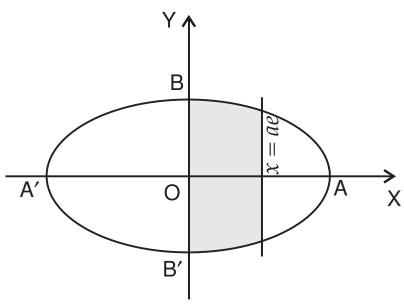

Fig. 11.17.

The required area is shown shaded in fig. 11.17.

Since the area is symmetrical about the x-axis,

required area = 2 (area of the region bounded by the given ellipse, x-axis and the lines $x = 0$ and $x = ae$)

$$= 2\int_0^{ae} y\, dx = 2\int_0^{ae} \frac{b}{a}\sqrt{a^2 - x^2}\, dx \qquad (\because y \geq 0 \text{ in the first quadrant})$$

$$= 2\ \frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$$

$$= \frac{b}{a}\left[\left(ae\sqrt{a^2-a^2e^2} + a^2\sin^{-1}e\right) - \left(0 + \frac{a^2}{2}\sin^{-1}0\right)\right]$$

$$= ab\left(e\sqrt{1-e^2} + \sin^{-1}e\right).$$

Example 12. *The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.*

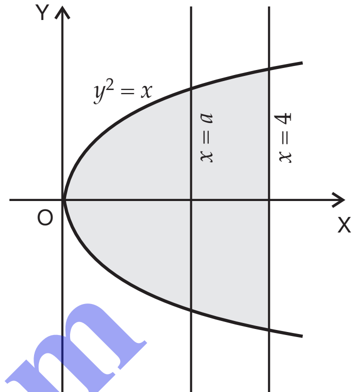

Fig. 11.18.

Solution. The given curve is $y^2 = x$ which represents a right hand parabola with vertex (0, 0).

The area bounded by the parabola and the line $x = 4$ is shown shaded in the fig. 11.18.

$$\text{This area} = 2\int_0^4 y\,dx = 2\int_0^4 \sqrt{x}\,dx$$

$$= 2\,.\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 = \frac{4}{3}(4^{3/2}-0) = \frac{4}{3}(8-0) = \frac{32}{3}.$$

Since the line $x = a$ divides this area into two equal parts, therefore,

$$2\int_0^a \sqrt{x}\,dx = \frac{1}{2}.\frac{32}{3} \Rightarrow \int_0^a \sqrt{x}\,dx = \frac{8}{3}$$

$$\Rightarrow \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^a = \frac{8}{3} \Rightarrow \frac{2}{3}(a^{3/2}-0) = \frac{8}{3}$$

$$\Rightarrow \quad a^{3/2} = 4 \Rightarrow a = 4^{2/3}$$

$$\Rightarrow \quad a = \sqrt[3]{16}.$$

Example 13. *Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $\frac{x}{a} + \frac{y}{b} = 1$.*

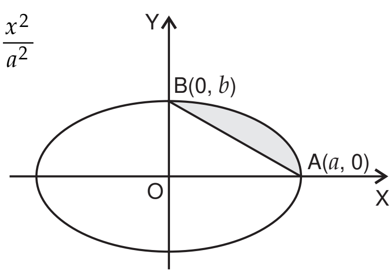

Fig. 11.19.

Solution. The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$$\Rightarrow \quad y = \frac{b}{a}\sqrt{a^2-x^2} \qquad (\because \text{ In first quadrant, } y \ge 0)$$

The given line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \quad \frac{y}{b} = 1 - \frac{x}{a} = \frac{a-x}{a}$$

$$\Rightarrow \quad y = \frac{b}{a}(a-x)$$

The area of the smaller region bounded by the given ellipse and the given line is shown shaded in the figure.

$$\text{Required area} = \int_0^a \left(\frac{b}{a}\sqrt{a^2-x^2} - \frac{b}{a}(a-x)\right)dx \qquad \text{(Article 11.1.1)}$$

$$= \frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - ax + \frac{x^2}{2}\right]_0^a$$

$$= \frac{b}{a}\left[\left(0 + \frac{a^2}{2}\sin^{-1}1 - a^2 + \frac{a^2}{2}\right) - \left(0 + \frac{a^2}{2}\sin^{-1}0 - 0 + 0\right)\right]$$

$$= \frac{b}{a}\left[\left(\frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{a^2}{2}\right) - 0\right] = \frac{1}{4}(\pi - 2)\,ab \text{ sq. units.}$$

Example 14. *Find the area of the region included between the curve $4y = 3x^2$ and the line $2y = 3x + 12$.*

Solution. The given curve is $4y = 3x^2$ ...(*i*)

It can be written as $y = \frac{3}{4}x^2$, which represents an upward parabola with vertex at (0, 0).

The given line is $3x - 2y + 12 = 0$

$\Rightarrow \quad y = \frac{3x+12}{2}$ ...(*ii*)

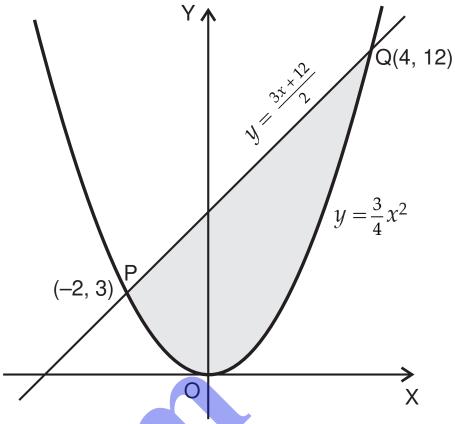

Fig. 11.20.

Solving (*i*) and (*ii*), we get

$$\frac{3x+12}{2} = \frac{3}{4}x^2$$

$\Rightarrow \quad 6x + 24 = 3x^2$

$\Rightarrow \quad x^2 - 2x - 8 = 0 \Rightarrow (x + 2)(x - 4) = 0$

$\Rightarrow \quad x = -2, x = 4.$

$\therefore$ The points of intersection are P (–2, 3) and Q (4, 12).

$\therefore$ Required area = area of the shaded region

$$= \int_{-2}^{4}\left(\frac{3x+12}{2} - \frac{3}{4}x^2\right)dx \qquad \text{[Article 11.1.1]}$$

$$= \left[\frac{3}{2}\cdot\frac{x^2}{2} + 6x - \frac{3}{4}\cdot\frac{x^3}{3}\right]_{-2}^{4} = \frac{1}{4}\left[3x^2 + 24x - x^3\right]_{-2}^{4}$$

$$= \frac{1}{4}\left[(48 + 96 - 64) - (12 - 48 + 8)\right]$$

$$= \frac{1}{4}\cdot 108 = 27 \text{ sq. units.}$$

Example 15. *Find the area enclosed by the parabola $y^2 = x$ and the line $y + x = 2$.*

Solution. The given parabola is $y^2 = x$ ...(*i*)

It represents a right hand parabola with vertex at (0, 0).

The given line is $y + x = 2$

i.e. $\quad x = 2 - y$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$$y^2 = 2 - y \Rightarrow y^2 + y - 2 = 0$$

$\Rightarrow \quad (y - 1)(y + 2) = 0 \Rightarrow y = 1, -2$

When $y = 1, x = 1$, when $y = -2, x = 4$

The points of intersection are P (1, 1) and Q (4, –2).

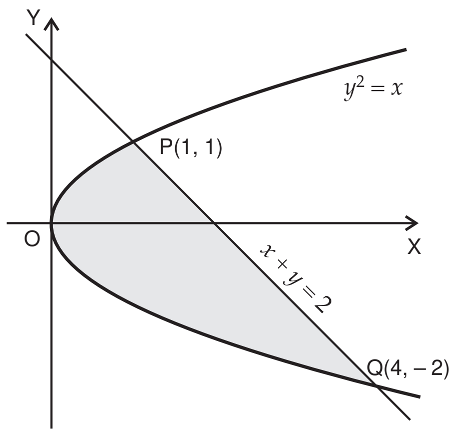

Fig. 11.21.

The required area = area of the shaded region

$$= \int_{-2}^{1}((2 - y) - y^2)\,dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 6 - \frac{8}{3} = 4\frac{1}{2} \text{ sq. units.}$$

Example 16. *Find the area bounded by the curve $y = 2x - x^2$ and the line $y = x$.* (*I.S.C. 2013*)

Solution. The given curve is $y = 2x - x^2$ ...(*i*)

It can be written as $y = -(x^2 - 2x + 1) + 1$

i.e. $(y - 1) = -(x - 1)^2$, which represents a downward parabola with vertex at (1, 1).

The given line is $y = x$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$x = 2x - x^2 \Rightarrow x^2 - x = 0$

$\Rightarrow$ $x = 0, 1.$

$\therefore$ The points of intersection are O (0, 0) and P (1, 1).

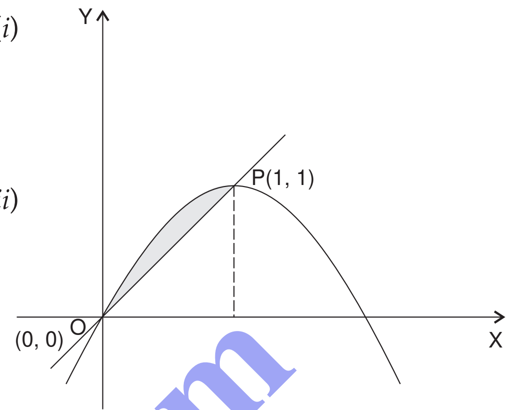

Fig. 11.22.

$\therefore$ Required area = area of the shaded region

$$= \int_0^1 ((2x - x^2) - x)\,dx = \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0) = \frac{1}{6} \text{ sq. units.}$$

Example 17. *Find the area enclosed by the curve $y = -x^2$ and the line $x + y + 2 = 0$.*

Solution. The given curve is $y = -x^2$ ...(*i*)

It represents a downward parabola with vertex O(0, 0).

The given line is $x + y + 2 = 0$

$\Rightarrow$ $y = -(x + 2)$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$-x^2 = -(x + 2) \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow$ $(x + 1)(x - 2) = 0 \Rightarrow x = -1, 2.$

When $x = -1$, $y = -1$ and when $x = 2$, $y = -4$.

$\therefore$ The points of intersection are P(–1, –1) and Q (2, –4).

The required area is shown shaded in fig. 11.23. We note that the required area lies below the x-axis, therefore,

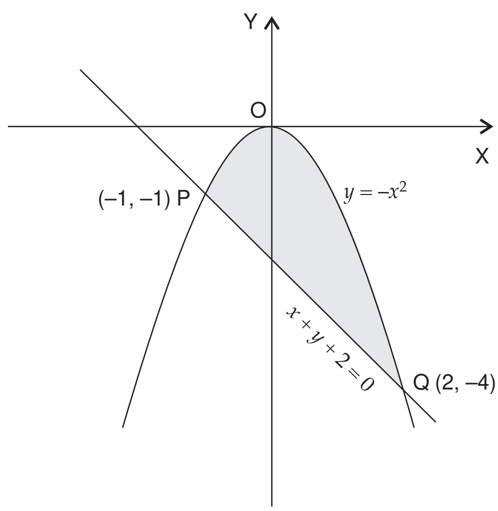

Fig. 11.23.

$$\text{required area} = \left|\int_{-1}^{2} (-(x+2) - (-x^2))\,dx\right|$$ [Article 11.1.1]

$$= \left|\left[-\left(\frac{x^2}{2} + 2x\right) + \frac{x^3}{3}\right]_{-1}^{2}\right|$$

$$= \left|\left(-6 + \frac{8}{3}\right) - \left(-\left(\frac{1}{2} - 2\right) - \frac{1}{3}\right)\right| = \frac{9}{2} \text{ sq. units.}$$

EXERCISE 11.1

1. (*i*) Find the area bounded by the curve $y = x^2$, the x-axis and the ordinates $x = 1$ and $x = 3$.

(*ii*) Find the area of the region bounded by $y^2 = x - 2$ and the lines $x = 4$ and $x = 6$.

(*iii*) Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

(*iv*) Find the area of the region bounded by $x^2 = y - 3$ and the lines $y = 4$ and $y = 6$.

2. Using integration, find the area of the region bounded between the line $x = 2$ and the parabola $y^2 = 8x$.

3. Using integration, find the area of the region bounded by the line $2y = -x + 8$, x-axis and the lines $x = 2$ and $x = 4$.

4. Make a rough sketch of the graph of the function $f(x) = 9 - x^2$, $0 \le x \le 3$ and determine the area enclosed between the curve and the axes.

5. Draw a rough sketch of the curve $y = \sqrt{3x+4}$ and find the area under the curve, above the x-axis and between $x = 0$ and $x = 4$.

6. Sketch the rough graph of $y = 4\sqrt{x-1}$, $1 \le x \le 3$ and compute the area between the curve, x-axis and the line $x = 3$.

7. Find the area enclosed between the curve $y = 2x - x^2$ and the x-axis.

8. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.

9. Find the area bounded by the curve $y = x^2 - 7x + 6$, the x-axis and the lines $x = 2$, $x = 6$.

10. Find the area of the region bounded by the curve $x = 4y - y^2$ and the y-axis.

(*I.S.C. 2012*)

11. Sketch the graph of the curve $y = \sqrt{x} + 1$, $0 \le x \le 4$ and determine the area of the region enclosed by the curve, x-axis and the lines $x = 0$ and $x = 4$.

12. Find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus-rectum.

13. (*i*) Find the area lying between the curve $y^2 = 4x$ and the line $y = 2x$.

(*ii*) Find the area enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.

14. Find the area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$.

15. Sketch the region $\{(x, y) ; 4x^2 + 9y^2 = 36\}$ and find its area, using integration.

16. Make a rough sketch of the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and find

(*i*) the area under the curve and above the x-axis.

(*ii*) the area enclosed by the curve.

17. Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

18. (*i*) Find the area of the smaller part enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

(*ii*) Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.

19. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the curve $x^2 + y^2 = 16$.

12. Find the area of the region enclosed by the curves $y = x^2$, $y = x^2 - 2x$ and the lines $x = 1$, $x = 3$.

13. Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$ as x varies from 0 to $\frac{\pi}{2}$ and find the area of the region enclosed by them and the x-axis.

14. Find the area enclosed by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$.

15. Find the area bounded by the curve $y = x^3$ and the line $y = x$.

ANSWERS

EXERCISE 11.1

1. (*i*) $\frac{26}{3}$ sq. units (*ii*) $\frac{8}{3}(4 - \sqrt{2})$ sq. units (*iii*) $\frac{8}{3}(4 - \sqrt{2})$ sq. units

(*iv*) $\frac{4}{3}(3\sqrt{3} - 1)$ sq. units.

2. $\frac{32}{3}$ sq. units. **3.** 5 sq. units. **4.** 18 sq. units.

5. $\frac{112}{9}$ sq. units **6.** $\frac{16\sqrt{2}}{3}$ sq. units. **7.** $\frac{4}{3}$ sq. units.

8. $\frac{4}{3}$ sq. units. **9.** $\frac{56}{3}$ sq. units. **10.** $\frac{32}{3}$ sq. units. **11.** $\frac{28}{3}$ sq. units.

12. $\frac{8}{3}a^2$ sq. units. **13.** (*i*) $\frac{1}{3}$ sq. units (*ii*) $\frac{8a^2}{3m^3}$ sq. units. **14.** π sq. units.

15. 6π sq. units. **16.** (*i*) 3π sq. units (*ii*) 6π sq. units.

17. πab sq. units. **18.** (*i*) $(\pi - 2)$ sq. units (*ii*) $\frac{a^2}{4}(\pi - 2)$ sq. units.

19. 2π sq. units. **20.** $\frac{3}{2}(\pi - 2)$ sq. units. **21.** (*i*) $\frac{1}{6}$ sq. units. (*ii*) $\frac{9}{8}$ sq. units.

22. 18 sq. units. **23.** $(\pi - 2)$ sq. units **25.** $\frac{2}{3}a^2$ sq. units.

26. (*i*) $\frac{16}{3}$ sq. units (*ii*) $\frac{16}{3}a^2$ sq. units.

27. (*i*) $\frac{23}{6}$ sq. units (*ii*) $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq. units (*iii*) $\frac{1}{3}$ sq. units.

28. $\frac{16}{3}$ sq. units. **29.** $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units. **30.** $\frac{13}{3}$ sq. units.

31. 4 sq. units. **32.** $\frac{\pi}{4}$ sq. units. **33.** 4 sq. units.

EXERCISE 11.2

1. 9; it represents the area below the graph, above the x-axis and bounded by the lines $x = -4$ and $x = 2$.

2. $\frac{16}{3}a$ sq. units. **3.** (*i*) $\frac{3}{2}$ sq. units (*ii*) 6 sq. units.

4. $(8\pi - \sqrt{3}) : (4\pi + \sqrt{3})$. **5.** 15 : 49.

6. $20\frac{5}{6}$ sq. units; $10\frac{2}{3}$ sq. units. **7.** $\left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right)$ sq. units.