19 Trigonometrical Identities

19.1 TRIGONOMETRICAL IDENTITY

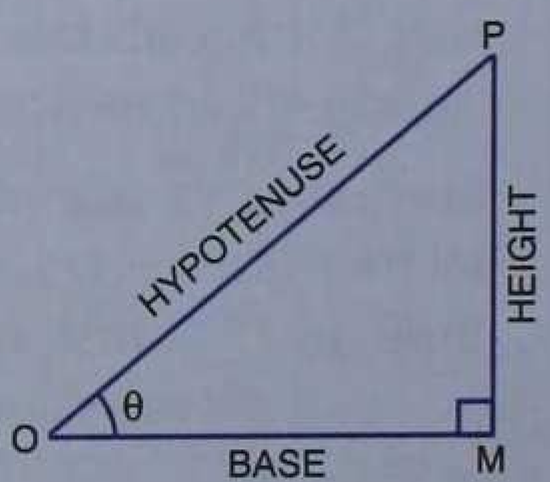
*An equation which is true for all values of the variable involved is called an **identity**.*

*An equation which involves trigonometric ratios of an angle and is true for all values of the angle is called a **trigonometrical identity**.*

In this chapter, we shall deal with the fundamental trigonometrical identities and will prove certain identities based upon them. However, before doing so, let us review trigonometric ratios.

19.1.1 Trigonometrical ratios

Let OMP be a right angled triangle at M and $\angle MOP = \theta$, then the *trigonometrical ratios* (abbreviated *t-ratios*) are defined as :



(1) $\frac{MP}{OP}$ is called **sine** of θ and is written as $\sin\theta$. Thus

$\sin\theta = \frac{MP}{OP}$.

(2) $\frac{OM}{OP}$ is called **cosine** of θ and is written as $\cos\theta$.

Thus $\cos\theta = \frac{OM}{OP}$.

(3) $\frac{MP}{OM}$ is called **tangent** of θ and is written as $\tan\theta$. Thus $\tan\theta = \frac{MP}{OM}$.

(4) $\frac{OM}{MP}$ is called **cotangent** of θ and is written as $\cot\theta$. Thus

$\cot\theta = \frac{OM}{MP}$.

(5) $\frac{OP}{OM}$ is called **secant** of θ and is written as $\sec\theta$. Thus

$\sec\theta = \frac{OP}{OM}$.

(6) $\frac{OP}{MP}$ is called **cosecant** of θ and is written as $\text{cosec}\,\theta$. Thus

$\text{cosec}\,\theta = \frac{OP}{MP}$.

In reference to $\angle MOP$ in Δ OMP, OM is called *base* or *adjoining side*, MP is called *height* or *opposite side* and OP is the *hypotenuse*. The *six trigonometrical ratios* can be defined as :

(1) $\sin\theta = \dfrac{\text{height}}{\text{hypotenuse}}$. (2) $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}}$.

(3) $\tan\theta = \dfrac{\text{height}}{\text{base}}$. (4) $\cot\theta = \dfrac{\text{base}}{\text{height}}$.

(5) $\sec\theta = \dfrac{\text{hypotenuse}}{\text{base}}$. (6) $\text{cosec}\,\theta = \dfrac{\text{hypotenuse}}{\text{height}}$.

Remarks

- In right-angled triangle OMP, $\angle MOP$ lies between 0° to 90° *i.e.* $\angle MOP$ is acute angle *i.e.* θ is acute and all the six trigonometrical ratios are positive.
- *Each trigonometrical ratio is a real number.*

19.1.2 Reciprocal relations

From the right-angled triangle OMP, we get

(1) $\sin\theta = \dfrac{MP}{OP}$ and $\text{cosec}\,\theta = \dfrac{OP}{MP}$

$\Rightarrow$ $\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$ and $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

$\Rightarrow$ *sin θ and cosec θ are reciprocals of each other.*

(2) $\cos\theta = \dfrac{OM}{OP}$ and $\sec\theta = \dfrac{OP}{OM}$

$\Rightarrow$ $\cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

$\Rightarrow$ *cos θ and sec θ are reciprocal of each other.*

(3) $\tan\theta = \dfrac{MP}{OM}$ and $\cot\theta = \dfrac{OM}{MP}$

$\Rightarrow$ $\tan\theta = \dfrac{1}{\cot\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$

$\Rightarrow$ *tan θ and cot θ are reciprocal of each other.*

Remark

(*i*) $\sin\theta \cdot \text{cosec}\,\theta = 1$
(*ii*) $\cos\theta \cdot \sec\theta = 1$
(*iii*) $\tan\theta \cdot \cot\theta = 1$.

19.1.3 Quotient relations

From the right-angled triangle OMP, we get

(1) $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{MP}{OP}}{\frac{OM}{OP}} = \dfrac{MP}{OP} \times \dfrac{OP}{OM} = \dfrac{MP}{OM} = \tan\theta,$

$\therefore\ \tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

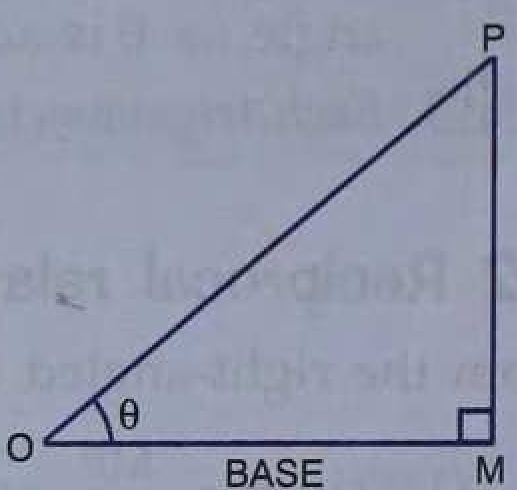
(2) $\dfrac{\cos\theta}{\sin\theta} = \dfrac{\frac{OM}{OP}}{\frac{MP}{OP}} = \dfrac{OM}{OP} \times \dfrac{OP}{MP} = \dfrac{OM}{MP} = \cot\theta$,

$\therefore\ \cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

19.2 FUNDAMENTAL IDENTITIES

1. $\sin^2\theta + \cos^2\theta = 1$.
2. $1 + \tan^2\theta = \sec^2\theta$.
3. $1 + \cot^2\theta = \text{cosec}^2\theta$.

Proof. Let OMP be a right-angled triangle at M and $\angle MOP = \theta$.



From the right-angled triangle OMP, by Pythagoras theorem, we get

$$MP^2 + OM^2 = OP^2. \qquad \ldots(i)$$

1. Dividing both sides of (*i*) by OP^2, we get

$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1$$

$\Rightarrow\ \sin^2\theta + \cos^2\theta = 1$.

2. Dividing both sides of (*i*) by OM^2, we get

$$\left(\frac{MP}{OM}\right)^2 + 1 = \left(\frac{OP}{OM}\right)^2$$

$\Rightarrow\ \tan^2\theta + 1 = \sec^2\theta \Rightarrow 1 + \tan^2\theta = \sec^2\theta$.

3. Dividing both sides of (*i*) by MP^2, we get

$$1 + \left(\frac{OM}{MP}\right)^2 = \left(\frac{OP}{MP}\right)^2$$

$\Rightarrow\ 1 + \cot^2\theta = \text{cosec}^2\theta$.

Note

$\sin^2\theta$ means $(\sin\theta)^2$ and $\sin^2\theta$ is read as *sine squared* θ. Similarly $\cos^2\theta$ means $(\cos\theta)^2$ etc.

Corollary 1. From identity 1, we deduce that

(*i*) $1 - \cos^2\theta = \sin^2\theta$ and (*ii*) $1 - \sin^2\theta = \cos^2\theta$.

Corollary 2. From identity 2, we deduce that

(*i*) $\sec^2\theta - 1 = \tan^2\theta$ and (*ii*) $\sec^2\theta - \tan^2\theta = 1$.

Corollary 3. From identity 3, we deduce that

(*i*) $\text{cosec}^2\theta - 1 = \cot^2\theta$ and (*ii*) $\text{cosec}^2\theta - \cot^2\theta = 1$.

19.2.1 Now, we take up some trigonometrical identities, which will be proved by using the results of Articles 19.1.2, 19.1.3 and the fundamental identities.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove the following identities :*

(i) $\tan^2\theta - \dfrac{1}{\cos^2\theta} + 1 = 0$

(ii) $\dfrac{1}{\sin\theta + \cos\theta} + \dfrac{1}{\sin\theta - \cos\theta} = \dfrac{2\sin\theta}{1 - 2\cos^2\theta}$. (2002)

Solution.

(i) $$\text{L.H.S.} = \tan^2\theta - \frac{1}{\cos^2\theta} + 1$$

$$= \tan^2\theta - \sec^2\theta + 1 \qquad \left(\because \frac{1}{\cos\theta} = \sec\theta\right)$$

$$= \tan^2\theta - (1 + \tan^2\theta) + 1 \qquad (\because \sec^2\theta = 1 + \tan^2\theta)$$

$$= \tan^2\theta - 1 - \tan^2\theta + 1 = 0 = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = \frac{(\sin\theta - \cos\theta) + (\sin\theta + \cos\theta)}{(\sin\theta + \cos\theta)\ (\sin\theta - \cos\theta)}$$

$$= \frac{2\sin\theta}{\sin^2\theta - \cos^2\theta} = \frac{2\sin\theta}{(1 - \cos^2\theta) - \cos^2\theta}$$

$$= \frac{2\sin\theta}{1 - 2\cos^2\theta} = \text{R.H.S.}$$

Example 2. *Prove the following identities :*

(i) $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$

(ii) $1 + \dfrac{\tan^2 A}{1 + \sec A} = \sec A$.

Solution.

(i) $$\text{L.H.S.} = \cot^2 A - \cos^2 A = \frac{\cos^2 A}{\sin^2 A} - \cos^2 A$$

$$= \cos^2 A\left(\frac{1}{\sin^2 A} - 1\right) = \cos^2 A\ (\text{cosec}^2 A - 1)$$

$$= \cos^2 A \cot^2 A = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = 1 + \frac{\tan^2 A}{1 + \sec A} = 1 + \frac{\sec^2 A - 1}{1 + \sec A}$$

$$= 1 + \frac{(\sec A - 1)(\sec A + 1)}{1 + \sec A}$$

$$= 1 + (\sec A - 1) = \sec A = \text{R.H.S.}$$

Example 3. *Prove the following identities :*

(i) $(1 + \tan^2\theta)(1 - \sin\theta)(1 + \sin\theta) = 1$

(ii) $\dfrac{\sin\theta}{1 - \cos\theta} = \text{cosec}\ \theta + \cot\ \theta$.

Solution.

(i) $$\text{L.H.S.} = (1 + \tan^2\theta)(1 - \sin\theta)(1 + \sin\theta)$$

$$= \sec^2\theta \,.\, (1 - \sin^2\theta) = \sec^2\theta \,.\, \cos^2\theta$$

$$= \frac{1}{\cos^2\theta} \,.\, \cos^2\theta = 1 = \text{R.H.S.}$$

(*ii*) L.H.S. $= \frac{\sin\theta}{1-\cos\theta} = \frac{\sin\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$

$= \frac{\sin\theta\,(1+\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta\,(1+\cos\theta)}{\sin^2\theta}$

$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$

$= \text{cosec}\,\theta + \cot\theta =$ R.H.S.

Example 4. *Prove the following identities :*

(*i*) $\frac{1}{\sin A + \cos A} + \frac{1}{\sin A - \cos A} = \frac{2\sin A}{1 - 2\cos^2 A}$ (2002)

(*ii*) $(1 + \tan^2 A) + \left(1 + \frac{1}{\tan^2 A}\right) = \frac{1}{\sin^2 A - \sin^4 A}$.

Solution.

(*i*) L.H.S. $= \frac{1}{\sin A + \cos A} + \frac{1}{\sin A - \cos A}$

$= \frac{(\sin A - \cos A) + (\sin A + \cos A)}{\sin^2 A - \cos^2 A} = \frac{2\sin A}{(1-\cos^2 A) - \cos^2 A}$

$= \frac{2\sin A}{1 - 2\cos^2 A} =$ R.H.S.

(*ii*) L.H.S. $= (1 + \tan^2 A) + \left(1 + \frac{1}{\tan^2 A}\right)$

$= \sec^2 A + (1 + \cot^2 A) = \sec^2 A + \text{cosec}^2 A$

$= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A}$

$= \frac{1}{(1-\sin^2 A)\sin^2 A} = \frac{1}{\sin^2 A - \sin^4 A}$.

Example 5. *Prove the following identities :*

(*i*) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

(*ii*) $\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$.

Solution.

(*i*) L.H.S. $= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}}$

$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$

$= \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$

$= \sec\theta + \tan\theta =$ R.H.S.

(*ii*) R.H.S. $= (\sec\theta - \tan\theta)^2 = \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2$

$= \left(\frac{1-\sin\theta}{\cos\theta}\right)^2 = \frac{(1-\sin\theta)^2}{\cos^2\theta}$

$$= \frac{(1 - \sin\theta)^2}{1 - \sin^2\theta} = \frac{(1 - \sin\theta)^2}{(1 - \sin\theta)(1 + \sin\theta)}$$

$$= \frac{1 - \sin\theta}{1 + \sin\theta} = \text{L.H.S.}$$

Example 6. *Prove the following identities :*

(i) $\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = 2 \operatorname{cosec} A$ (2009, 04)

(ii) $\frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A} = \sin A + \cos A.$ (2003)

Solution.

(i) $$\text{L.H.S.} = \frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = \frac{\sin^2 A + (1 + \cos A)^2}{(1 + \cos A)\sin A}$$

$$= \frac{\sin^2 A + 1 + 2\cos A + \cos^2 A}{(1 + \cos A)\sin A} = \frac{(\sin^2 A + \cos^2 A) + 1 + 2\cos A}{(1 + \cos A)\sin A}$$

$$= \frac{1 + 1 + 2\cos A}{(1 + \cos A)\sin A} = \frac{2(1 + \cos A)}{(1 + \cos A)\sin A}$$

$$= \frac{2}{\sin A} = 2 \operatorname{cosec} A = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = \frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A}$$

$$= \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin A}{1 - \frac{\cos A}{\sin A}} = \frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin^2 A}{\sin A - \cos A}$$

$$= \frac{\cos^2 A}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A} = \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A}$$

$$= \frac{(\cos A + \sin A)(\cos A - \sin A)}{\cos A - \sin A} = \cos A + \sin A = \text{R.H.S.}$$

Example 7. *Prove the following identities :*

(i) $\tan^4 A + \tan^2 A = \sec^4 A - \sec^2 A$

(ii) $\frac{\cos^2 A + \tan^2 A - 1}{\sin^2 A} = \tan^2 A.$

Solution.

(i) $$\text{L.H.S.} = \tan^4 A + \tan^2 A$$
$$= \tan^2 A (\tan^2 A + 1)$$
$$= (\sec^2 A - 1)\sec^2 A \qquad (\because 1 + \tan^2 A = \sec^2 A)$$
$$= \sec^4 A - \sec^2 A = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = \frac{\cos^2 A + \tan^2 A - 1}{\sin^2 A} = \frac{\tan^2 A - (1 - \cos^2 A)}{\sin^2 A}$$

$$= \frac{\tan^2 A - \sin^2 A}{\sin^2 A} = \frac{\tan^2 A}{\sin^2 A} - \frac{\sin^2 A}{\sin^2 A}$$

$$= \frac{\sin^2 A}{\cos^2 A} \cdot \frac{1}{\sin^2 A} - 1 = \frac{1}{\cos^2 A} - 1$$

$$= \sec^2 A - 1 = \tan^2 A = \text{R.H.S.}$$

Example 8. *Prove the following identities :*

(i) sin θ (1 + tan θ) + cos θ (1 + cot θ) = sec θ + cosec θ

(ii) (1 + cot θ – cosec θ) (1 + tan θ + sec θ) = 2.

Solution.

(*i*) L.H.S. $= \sin\theta\,(1 + \tan\theta) + \cos\theta\,(1 + \cot\theta)$

$$= \sin\theta\left(1+\frac{\sin\theta}{\cos\theta}\right)+\cos\theta\left(1+\frac{\cos\theta}{\sin\theta}\right)$$

$$= \sin\theta\left(\frac{\cos\theta+\sin\theta}{\cos\theta}\right)+\cos\theta\left(\frac{\sin\theta+\cos\theta}{\sin\theta}\right)$$

$$= (\cos\theta+\sin\theta)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$= (\cos\theta+\sin\theta)\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right) = (\cos\theta+\sin\theta)\left(\frac{1}{\cos\theta\sin\theta}\right)$$

$$= \frac{\cos\theta}{\cos\theta\sin\theta}+\frac{\sin\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta}+\frac{1}{\cos\theta}$$

$= \text{cosec}\,\theta + \sec\theta =$ R.H.S.

(*ii*) L.H.S. $= (1 + \cot\theta - \text{cosec}\,\theta)(1 + \tan\theta + \sec\theta)$

$$= \left(1+\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)\left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)$$

$$= \left(\frac{\sin\theta+\cos\theta-1}{\sin\theta}\right)\left(\frac{\cos\theta+\sin\theta+1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta+\cos\theta)^2-1}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta} = \frac{1+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{R.H.S.}$$

Example 9. *Prove the following identities :*

(*i*) $\dfrac{\sec A+\tan A}{\text{cosec}\,A+\cot A} = \dfrac{\text{cosec}\,A-\cot A}{\sec A-\tan A}$

(*ii*) $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A} = 1 + \tan A + \cot A.$

Solution.

(*i*) $\dfrac{\sec A+\tan A}{\text{cosec}\,A+\cot A} = \dfrac{\text{cosec}\,A-\cot A}{\sec A-\tan A}$ is true

if $(\sec A + \tan A)(\sec A - \tan A) = (\text{cosec}\,A + \cot A)(\text{cosec}\,A - \cot A)$ is true

i.e. if $\sec^2 A - \tan^2 A = \text{cosec}^2 A - \cot^2 A$ is true

i.e. if $(1 + \tan^2 A) - \tan^2 A = (1 + \cot^2 A) - \cot^2 A$ is true

i.e. if 1 = 1 is true, which is true.

(*ii*) L.H.S. $= \dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A} = \dfrac{\frac{\sin A}{\cos A}}{1-\frac{\cos A}{\sin A}}+\dfrac{\frac{\cos A}{\sin A}}{1-\frac{\sin A}{\cos A}}$

$$= \frac{\sin^2 A}{\cos A(\sin A-\cos A)}+\frac{\cos^2 A}{\sin A(\cos A-\sin A)}$$

$$= \frac{1}{\sin A - \cos A}\left(\frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A}\right)$$

$$= \frac{1}{\sin A - \cos A} \cdot \frac{\sin^3 A - \cos^3 A}{\cos A \sin A}$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \sin A \cos A + \cos^2 A)}{(\sin A - \cos A) \sin A \cos A}$$

$$= \frac{\sin^2 A + \sin A \cos A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{\sin^2 A}{\sin A \cos A} + \frac{\sin A \cos A}{\sin A \cos A} + \frac{\cos^2 A}{\sin A \cos A}$$

$$= \frac{\sin A}{\cos A} + 1 + \frac{\cos A}{\sin A} = \tan A + 1 + \cot A$$

$$= 1 + \tan A + \cot A = \text{R.H.S.}$$

Example 10. *Prove the following identities :*

(i) $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}$

(ii) $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\operatorname{cosec}^2 A} - \dfrac{\operatorname{cosec} A}{\sec^2 A}$.

Solution.

(*i*) L.H.S. $= \dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1}$

$$= \frac{(\tan A + \sec A) - (\sec^2 A - \tan^2 A)}{\tan A - \sec A + 1} \qquad (\because \sec^2 A - \tan^2 A = 1)$$

$$= \frac{(\tan A + \sec A) - (\sec A + \tan A)(\sec A - \tan A)}{\tan A - \sec A + 1}$$

$$= \frac{(\tan A + \sec A)(1 - \sec A + \tan A)}{\tan A - \sec A + 1}$$

$$= \tan A + \sec A = \frac{\sin A}{\cos A} + \frac{1}{\cos A}$$

$$= \frac{\sin A + 1}{\cos A} = \frac{1 + \sin A}{\cos A} = \text{R.H.S.}$$

(*ii*) L.H.S. $= (1 + \cot A + \tan A)(\sin A - \cos A)$

$$= \left(1 + \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}\right)(\sin A - \cos A)$$

$$= \left(\frac{\sin A \cos A + \cos^2 A + \sin^2 A}{\sin A \cos A}\right)(\sin A - \cos A)$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \cos^2 A + \sin A \cos A)}{\sin A \cos A}$$

$$= \frac{\sin^3 A - \cos^3 A}{\sin A \cos A}$$

$$\text{R.H.S.} = \frac{\sec A}{\text{cosec}^2 A} - \frac{\text{cosec } A}{\sec^2 A} = \frac{\frac{1}{\cos A}}{\frac{1}{\sin^2 A}} - \frac{\frac{1}{\sin A}}{\frac{1}{\cos^2 A}}$$

$$= \frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A} = \frac{\sin^3 A - \cos^3 A}{\cos A \sin A}$$

$\Rightarrow$ L.H.S. = R.H.S.

Example 11. *Prove the following identities :*

(i) $\tan^2 \theta + \cot^2 \theta + 2 = \sec^2 \theta \text{ cosec}^2 \theta$

(ii) $\sqrt{\sec^2 \theta + \text{cosec}^2 \theta} = \tan \theta + \cot \theta.$

Solution.

(i) L.H.S. $= \tan^2 \theta + \cot^2 \theta + 2$

$= (1 + \tan^2 \theta) + (1 + \cot^2 \theta)$

$= \sec^2 \theta + \text{cosec}^2 \theta$

$= \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin^2 \theta}$

$= \frac{1}{\cos^2 \theta \sin^2 \theta} = \frac{1}{\cos^2 \theta} \cdot \frac{1}{\sin^2 \theta}$

$= \sec^2 \theta \text{ cosec}^2 \theta$ = R.H.S.

(ii) L.H.S. $= \sqrt{\sec^2 \theta + \text{cosec}^2 \theta}$

$= \sqrt{(1 + \tan^2 \theta) + (1 + \cot^2 \theta)} = \sqrt{\tan^2 \theta + \cot^2 \theta + 2}$

$= \sqrt{\tan^2 \theta + \cot^2 \theta + 2 \tan \theta \cot \theta}$ $\quad (\because \tan \theta \cot \theta = 1)$

$= \sqrt{(\tan \theta + \cot \theta)^2} = \tan \theta + \cot \theta$ = R.H.S.

Example 12. *Prove the following identities :*

(i) $\sin^4 \theta + \cos^4 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta$

(ii) $\frac{1}{\text{cosec } \theta - \cot \theta} - \frac{1}{\sin \theta} = \frac{1}{\sin \theta} - \frac{1}{\text{cosec } \theta + \cot \theta}.$

Solution.

(i) L.H.S. $= \sin^4 \theta + \cos^4 \theta = (\sin^2 \theta)^2 + (\cos^2 \theta)^2$

$= (\sin^2 \theta + \cos^2 \theta)^2 - 2.\sin^2 \theta \cos^2 \theta$ $\quad [\because a^2 + b^2 = (a + b)^2 - 2ab]$

$= 1^2 - 2 \sin^2 \theta \cos^2 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta$ = R.H.S.

(ii) $\frac{1}{\text{cosec } \theta - \cot \theta} - \frac{1}{\sin \theta} = \frac{1}{\sin \theta} - \frac{1}{\text{cosec } \theta + \cot \theta}$ is true

if $\frac{1}{\text{cosec } \theta - \cot \theta} + \frac{1}{\text{cosec } \theta + \cot \theta} = \frac{1}{\sin \theta} + \frac{1}{\sin \theta}$ is true

i.e. if $\frac{(\text{cosec } \theta + \cot \theta) + (\text{cosec } \theta - \cot \theta)}{(\text{cosec } \theta - \cot \theta)(\text{cosec } \theta + \cot \theta)} = \frac{2}{\sin \theta}$ is true

i.e. if $\frac{2 \text{ cosec } \theta}{\text{cosec}^2 \theta - \cot^2 \theta} = 2 \text{ cosec } \theta$ is true

i.e. if $\frac{2 \text{ cosec } \theta}{1} = 2 \text{ cosec } \theta$ is true, $\quad (\because \text{cosec}^2 \theta - \cot^2 \theta = 1)$

which is true.

Example 13. *Prove the following identities :*

(i) $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\left(\dfrac{1-\tan\theta}{1-\cot\theta}\right)^2=\tan^2\theta$

(ii) $\dfrac{\sin A-\sin B}{\cos A+\cos B}+\dfrac{\cos A-\cos B}{\sin A+\sin B}=0.$

Solution.

(i) $$\frac{1+\tan^2\theta}{1+\cot^2\theta}=\frac{\sec^2\theta}{\operatorname{cosec}^2\theta}=\frac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}}$$

$$=\frac{1}{\cos^2\theta}\times\frac{\sin^2\theta}{1}=\left(\frac{\sin\theta}{\cos\theta}\right)^2=\tan^2\theta.$$

Also $$\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2=\left(\frac{1-\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}}\right)^2=\left(\frac{\frac{\cos\theta-\sin\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\sin\theta}}\right)^2$$

$$=\left(-\frac{\sin\theta-\cos\theta}{\cos\theta}\times\frac{\sin\theta}{\sin\theta-\cos\theta}\right)^2=\left(-\frac{\sin\theta}{\cos\theta}\right)^2$$

$$=(-\tan\theta)^2=\tan^2\theta.$$

$$\therefore\quad\frac{1+\tan^2\theta}{1+\cot^2\theta}=\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2=\tan^2\theta.$$

(ii) $$\text{L.H.S.}=\frac{\sin A-\sin B}{\cos A+\cos B}+\frac{\cos A-\cos B}{\sin A+\sin B}$$

$$=\frac{(\sin A-\sin B)(\sin A+\sin B)+(\cos A-\cos B)(\cos A+\cos B)}{(\cos A+\cos B)(\sin A+\sin B)}$$

$$=\frac{(\sin^2 A-\sin^2 B)+(\cos^2 A-\cos^2 B)}{(\cos A+\cos B)(\sin A+\sin B)}$$

$$=\frac{(\sin^2 A+\cos^2 A)-(\sin^2 B+\cos^2 B)}{(\cos A+\cos B)(\sin A+\sin B)}$$

$$=\frac{1-1}{(\cos A+\cos B)(\sin A+\sin B)}$$

$$=0=\text{R.H.S.}$$

Example 14. *Eliminate θ between the equations :*

$$x=a\cos\theta+b\sin\theta,\ y=a\sin\theta-b\cos\theta.$$

Solution. Given $x=a\cos\theta+b\sin\theta$ ...(i)

and $y=a\sin\theta-b\cos\theta$ ...(ii)

Squaring (i) and (ii) and adding, we get

$$x^2+y^2=a^2\cos^2\theta+b^2\sin^2\theta+2ab\sin\theta\cos\theta+a^2\sin^2\theta+b^2\cos^2\theta-2ab\sin\theta\cos\theta$$

$$=a^2(\cos^2\theta+\sin^2\theta)+b^2(\sin^2\theta+\cos^2\theta)$$

$$=a^2\,.\,1+b^2\,.\,1$$

$\Rightarrow$ $x^2+y^2=a^2+b^2$, which is the required eliminant.

Example 15. *Eliminate θ between the equations :*

$$\text{cosec } \theta + \cot \theta = p, \text{ cosec } \theta - \cot \theta = q.$$

Solution. Given $\quad \text{cosec } \theta + \cot \theta = p \quad \ldots(i)$

and $\quad \text{cosec } \theta - \cot \theta = q \quad \ldots(ii)$

Multiplying (*i*) and (*ii*), we get

$(\text{cosec } \theta + \cot \theta)(\text{cosec } \theta - \cot \theta) = pq$

$\Rightarrow \text{cosec}^2 \theta - \cot^2 \theta = pq$

$\Rightarrow 1 = pq$, which is the required eliminant.

Example 16. *If $\cos \theta + \sin \theta = m$ and $\sec \theta + \text{cosec } \theta = n$, prove that $n(m^2 - 1) = 2m$.*

Solution. Given $\cos \theta + \sin \theta = m \quad \ldots(i)$

and $\quad \sec \theta + \text{cosec } \theta = n \quad \ldots(ii)$

From (*ii*), we get $\dfrac{1}{\cos \theta} + \dfrac{1}{\sin \theta} = n$

$\Rightarrow \sin \theta + \cos \theta = n \sin \theta \cos \theta$

$\Rightarrow m = n \sin \theta \cos \theta \quad$ (Using (*i*))

$\Rightarrow \sin \theta \cos \theta = \dfrac{m}{n} \quad \ldots(iii)$

On squaring (*i*), we get $\cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta = m^2$

$\Rightarrow 1 + 2.\dfrac{m}{n} = m^2 \quad$ (Using (*iii*))

$\Rightarrow \dfrac{2m}{n} = m^2 - 1 \Rightarrow n(m^2 - 1) = 2m$, as required.

Example 17. *Without using tables, find the value of :*

$$14 \sin 30° + 6 \cos 60° - 5 \tan 45°. \quad (2004)$$

Solution. $14 \sin 30° + 6 \cos 60° - 5 \tan 45°$

$= 14 \times \dfrac{1}{2} + 6 \times \dfrac{1}{2} - 5 \times 1$

$= 7 + 3 - 5 = 5.$

Example 18. *Without using mathematical tables, find the value of x if :*

$$\cos x = \cos 60° \cos 30° + \sin 60° \sin 30°. \quad (2005)$$

Solution. Given $\cos x = \cos 60° \cos 30° + \sin 60° \sin 30°$

$\Rightarrow \cos x = \dfrac{1}{2}.\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{2}.\dfrac{1}{2}$

$\Rightarrow \cos x = \dfrac{\sqrt{3} + \sqrt{3}}{4} = \dfrac{2\sqrt{3}}{4}$

$\Rightarrow \cos x = \dfrac{\sqrt{3}}{2} \Rightarrow x = 30°.$

Example 19. *When $0° < \theta < 90°$, solve the following equations :*

(*i*) $\dfrac{\cos \theta}{1 - \sin \theta} + \dfrac{\cos \theta}{1 + \sin \theta} = 4$

(*ii*) $\dfrac{\cos^2 \theta - 3 \cos \theta + 2}{\sin^2 \theta} = 1.$

Solution.

(*i*) Given $\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}=4$

$\Rightarrow \quad \frac{\cos\theta(1+\sin\theta)+\cos\theta(1-\sin\theta)}{1-\sin^2\theta}=4$

$\Rightarrow \quad \frac{2\cos\theta}{\cos^2\theta}=4 \Rightarrow \cos\theta=\frac{1}{2}$

$\Rightarrow \quad \theta=60°.$

(*ii*) Given $\frac{\cos^2\theta-3\cos\theta+2}{\sin^2\theta}=1$

$\Rightarrow \quad \cos^2\theta-3\cos\theta+2=\sin^2\theta$

$\Rightarrow \quad \cos^2\theta-3\cos\theta+2=1-\cos^2\theta$

$\Rightarrow \quad 2\cos^2\theta-3\cos\theta+1=0$

$\Rightarrow \quad (2\cos\theta-1)(\cos\theta-1)=0$

$\Rightarrow \quad \cos\theta=\frac{1}{2}$ or $\cos\theta=1$

$\Rightarrow \quad \theta=60°$ or $\theta=0°$ but $0°<\theta<90°$

$\Rightarrow \quad \theta=60°.$

Exercise 19.1

Prove the following trigonometrical identities (1 to 20) :

1. (*i*) $\sin\theta\cot\theta+\sin\theta\,\mathrm{cosec}\,\theta=1+\cos\theta$

(*ii*) $\frac{1}{1+\tan^2\theta}+\frac{1}{1+\cot^2\theta}=1$ (*iii*) $\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta=1.$

2. (*i*) $\cot^2 A-\frac{1}{\sin^2 A}+1=0$ (*ii*) $\sec A(1-\sin A)(\sec A+\tan A)=1.$

3. (*i*) $\frac{1}{1+\cos A}+\frac{1}{1-\cos A}=2\,\mathrm{cosec}^2 A$ (*ii*) $\frac{1}{\sec A+\tan A}+\frac{1}{\sec A-\tan A}=2\sec A.$

4. (*i*) $\frac{\sin A}{1+\cos A}=\frac{1-\cos A}{\sin A}.$ (*ii*) $\frac{1-\tan^2 A}{\cot^2 A-1}=\tan^2 A$

(*iii*) $\frac{\sin A}{1+\cos A}=\mathrm{cosec}\,A-\cot A.$ (*2008*)

5. (*i*) $\frac{\sec A-1}{\sec A+1}=\frac{1-\cos A}{1+\cos A}$ (*2007*) (*ii*) $\frac{\tan^2\theta}{(\sec\theta-1)^2}=\frac{1+\cos\theta}{1-\cos\theta}$ (*2012*)

(*iii*) $(1+\tan A)^2+(1-\tan A)^2=2\sec^2 A$ (*2005*)

(*iv*) $\sec^2 A+\mathrm{cosec}^2 A=\sec^2 A\,\mathrm{cosec}^2 A.$

6. (*i*) $\frac{1+\sin A}{\cos A}+\frac{\cos A}{1+\sin A}=2\sec A$ (*ii*) $\frac{\tan A}{\sec A-1}+\frac{\tan A}{\sec A+1}=2\,\mathrm{cosec}\,A.$

7. (*i*) $\frac{\mathrm{cosec}\,A}{\mathrm{cosec}\,A-1}+\frac{\mathrm{cosec}\,A}{\mathrm{cosec}\,A+1}=2\sec^2 A$ (*ii*) $\cot A-\tan A=\frac{2\cos^2 A-1}{\sin A\cos A}$

(*iii*) $\frac{\cot A-1}{2-\sec^2 A}=\frac{\cot A}{1+\tan A}.$

8. (*i*) $\tan^2\theta-\sin^2\theta=\tan^2\theta\sin^2\theta$ (*ii*) $\frac{\cos\theta}{1-\tan\theta}-\frac{\sin^2\theta}{\cos\theta-\sin\theta}=\cos\theta+\sin\theta.$

9. (i) $1 - \frac{\cos^2 \theta}{1 + \sin \theta} = \sin \theta$ (2001)

(ii) $\sqrt{\frac{1 - \cos A}{1 + \cos A}} = \frac{\sin A}{1 + \cos A}$. (2013)

10. (i) $\sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} = \text{cosec } \theta + \cot \theta$ (ii) $\frac{\sin \theta \tan \theta}{1 - \cos \theta} = 1 + \sec \theta$. (2006)

11. (i) $\frac{1 - \cos A}{1 + \cos A} = (\text{cosec } A - \cot A)^2$ (ii) $\frac{\cos A \cot A}{1 - \sin A} = 1 + \text{cosec } A$.

12. (i) $\sin^4 \theta - \cos^4 \theta = 1 - 2 \cos^2 \theta$ (ii) $\sec^4 \theta - \tan^4 \theta = 1 + 2 \tan^2 \theta$.

13. $\sin^6 \theta + \cos^6 \theta = 1 - 3 \sin^2 \theta \cos^2 \theta$.

Hint

L.H.S. $= (\sin^2 \theta)^3 + (\cos^2 \theta)^3$

$= (\sin^2 \theta + \cos^2 \theta)^3 - 3 \sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta)$,

$\because a^3 + b^3 = (a + b)^3 - 3ab(a + b)$.

14. (i) $\frac{1 + \tan A}{\sin A} + \frac{1 + \cot A}{\cos A} = 2(\sec A + \text{cosec } A)$

(ii) $(\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2 = \tan^2 A + \cot^2 A + 7$.

15. (i) $\text{cosec}^6 A - \cot^6 A = 3 \cot^2 A \text{ cosec}^2 A + 1$

(ii) $\sec^6 A - \tan^6 A = 1 + 3 \tan^2 A + 3 \tan^4 A$.

16. (i) $\frac{\cot \theta + \text{cosec } \theta - 1}{\cot \theta - \text{cosec } \theta + 1} = \frac{1 + \cos \theta}{\sin \theta}$ (ii) $\frac{\sin \theta}{\cot \theta + \text{cosec } \theta} = 2 + \frac{\sin \theta}{\cot \theta - \text{cosec } \theta}$.

17. (i) $(\sin \theta + \cos \theta)(\sec \theta + \text{cosec } \theta) = 2 + \sec \theta \text{ cosec } \theta$

(ii) $(\text{cosec } A - \sin A)(\sec A - \cos A) \sec^2 A = \tan A$ (2011)

(iii) $(\text{cosec } \theta - \sin \theta)(\sec \theta - \cos \theta)(\tan \theta + \cot \theta) = 1$.

18. (i) $\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$

(ii) $\frac{\tan^2 A}{1 + \tan^2 A} + \frac{\cot^2 A}{1 + \cot^2 A} = 1$.

19. (i) $\frac{1}{\sec A + \tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A - \tan A}$

(ii) $\frac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B$

(iii) $\tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B} = \sec^2 A - \sec^2 B$.

20. (i) $\frac{\sec \theta - 1}{\sec \theta + 1} = (\cot \theta - \text{cosec } \theta)^2$ (ii) $\frac{\tan \theta + \sin \theta}{\tan \theta - \sin \theta} = \frac{\sec \theta + 1}{\sec \theta - 1}$.

Eliminate θ between the equations (21 to 23) :

21. $x = a \sec \theta$, $y = b \tan \theta$.

22. $x = h + a \cos \theta$, $y = k + b \sin \theta$.

23. $\sec \theta + \tan \theta = m$, $\sec \theta - \tan \theta = n$.

24. If $2 \sin A - 1 = 0$, show that $\sin 3A = 3 \sin A - 4 \sin^3 A$. (2001)

When $0° < \theta < 90°$, solve the following equations (25 to 27) :

25. (*i*) $2 \sin^2 \theta = \frac{1}{2}$ (*ii*) $2 \cos 3\theta = 1$.

26. (*i*) $4 \cos^2 \theta - 3 = 0$ (*ii*) $\sin^2 \theta - \frac{1}{2} \sin \theta = 0$.

27. (*i*) $2 \cos^2 \theta + \sin \theta - 2 = 0$ (*ii*) $3 \cos \theta = 2 \sin^2 \theta$.

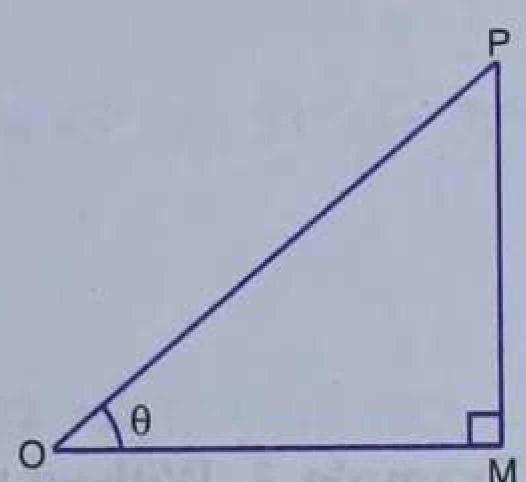
19.3 TRIGONOMETRICAL RATIOS OF COMPLEMENTARY ANGLES

Complementary angles. *Two angles are called* ***complementary*** *if the sum of their measure is 90°.*

Trigonometrical ratios of complementary angles

Let OMP be a right angled triangle at M and $\angle MOP = \theta$, then

$$\angle MPO = 180° - (\angle OMP + \angle MOP)$$
$$= 180° - (90° + \theta) = 90° - \theta.$$



(1) $\sin (90° - \theta) = \frac{OM}{OP} = \cos \theta$.

(2) $\cos (90° - \theta) = \frac{MP}{OP} = \sin \theta$.

(3) $\tan (90° - \theta) = \frac{OM}{MP} = \cot \theta$.

(4) $\cot (90° - \theta) = \frac{MP}{OM} = \tan \theta$.

(5) $\sec (90° - \theta) = \frac{OP}{MP} = \text{cosec}\, \theta$.

(6) $\text{cosec}\, (90° - \theta) = \frac{OP}{OM} = \sec \theta$.

Hence, **$\sin (90° - \theta) = \cos \theta$**

$\cos (90° - \theta) = \sin \theta$

$\tan (90° - \theta) = \cot \theta$

$\cot (90° - \theta) = \tan \theta$

$\sec (90° - \theta) = \text{cosec}\, \theta$

$\text{cosec}\, (90° - \theta) = \sec \theta$.

ILLUSTRATIVE EXAMPLES

Example 1. *Without using trigonometric tables, evaluate :*

(*i*) $\frac{\sin 23°}{\cos 67°}$ (*ii*) $\frac{\tan 65°}{\cot 25°}$ (*iii*) $\frac{\text{cosec}\, 31°}{\sec 59°}$.

Solution.

(*i*) $\frac{\sin 23°}{\cos 67°} = \frac{\sin 23°}{\cos (90° - 23°)} = \frac{\sin 23°}{\sin 23°}$ $(\because \cos (90° - \theta) = \sin \theta)$

$= 1$

(*ii*) $\frac{\tan 65°}{\cot 25°} = \frac{\tan 65°}{\cot (90° - 65°)} = \frac{\tan 65°}{\tan 65°}$ $(\because \cot (90° - \theta) = \tan \theta)$

$= 1$

(*iii*) $\frac{\text{cosec } 31°}{\sec 59°} = \frac{\text{cosec } (90° - 59°)}{\sec 59°} = \frac{\sec 59°}{\sec 59°}$ $(\because \text{cosec } (90° - \theta) = \sec \theta)$

$= 1.$

Example 2. *Without using trigonometric tables, evaluate :*

(i) **sin 18° – cos 72°**

(ii) sin 35° sin 55° – cos 35° cos 55°

(iii) 3 cos 80° cosec 10° + 2 sin 59° sec 31°. ***(2013)***

Solution.

(*i*) $\sin 18° - \cos 72° = \sin 18° - \cos (90° - 18°)$

$= \sin 18° - \sin 18° = 0.$

(*ii*) $\sin 35° \sin 55° - \cos 35° \cos 55°$

$= \sin 35° \sin (90° - 35°) - \cos 35° \cos (90° - 35°)$

$= \sin 35° \cos 35° - \cos 35° \sin 35°$

$= 0.$

(*iii*) $3 \cos 80° \text{ cosec } 10° + 2 \sin 59° \sec 31°$

$= 3 \cos 80° \text{ cosec } (90° - 80°) + 2 \sin 59° \sec (90° - 59°)$

$= 3 \cos 80° \sec 80° + 2 \sin 59° \text{ cosec } 59°$

$= 3 \cos 80° \times \frac{1}{\cos 80°} + 2 \sin 59° \times \frac{1}{\sin 59°}$

$= 3 \times 1 + 2 \times 1 = 3 + 2 = 5.$

Example 3. *Without using trigonometric tables, evaluate :*

(i) $\frac{2 \tan 53°}{\cot 37°} - \frac{\cot 80°}{\tan 10°}$ ***(2006)***

(ii) $\frac{\sin 80°}{\cos 10°} + \sin 59° \sec 31°.$ ***(2007)***

Solution.

(*i*) $\frac{2 \tan 53°}{\cot 37°} - \frac{\cot 80°}{\tan 10°} = \frac{2 \tan (90° - 37°)}{\cot 37°} - \frac{\cot (90° - 10°)}{\tan 10°}$

$= \frac{2 \cot 37°}{\cot 37°} - \frac{\tan 10°}{\tan 10°}$

$= 2 \times 1 - 1 = 2 - 1 = 1.$

(*ii*) $\frac{\sin 80°}{\cos 10°} + \sin 59° \sec 31° = \frac{\sin (90° - 10°)}{\cos 10°} + \sin (90° - 31°) \sec 31°$

$= \frac{\cos 10°}{\cos 10°} + \cos 31° \sec 31°$

$= 1 + \cos 31° \times \frac{1}{\cos 31°}$

$= 1 + 1 = 2.$

Example 4. *Without using trigonometric tables, evaluate :*

(i) $\sin^2 38° + \sin^2 52°$

(ii) $\left(\frac{\sin 47°}{\cos 43°}\right)^2 + \left(\frac{\cos 43°}{\sin 47°}\right)^2 - 4 \cos^2 45°.$

Solution.

(*i*) $\sin^2 38° + \sin^2 52° = \sin^2 38° + \sin^2 (90° - 38°)$

$= \sin^2 38° + \cos^2 38°$

$= 1.$ $\quad (\because \sin^2\theta + \cos^2\theta = 1)$

(*ii*) $\left(\dfrac{\sin 47°}{\cos 43°}\right)^2 + \left(\dfrac{\cos 43°}{\sin 47°}\right)^2 - 4\cos^2 45°$

$= \left(\dfrac{\sin (90° - 43°)}{\cos 43°}\right)^2 + \left(\dfrac{\cos 43°}{\sin (90° - 43°)}\right)^2 - 4 . \left(\dfrac{1}{\sqrt{2}}\right)^2$ $\quad \left(\because \cos 45° = \dfrac{1}{\sqrt{2}}\right)$

$= \left(\dfrac{\cos 43°}{\cos 43°}\right)^2 + \left(\dfrac{\cos 43°}{\cos 43°}\right)^2 - 4 . \dfrac{1}{2}$

$= 1^2 + 1^2 - 2 = 1 + 1 - 2 = 0.$

Example 5. *Without using trigonometrical tables, evaluate :*

(*i*) $\frac{2}{3}\text{cosec}^2\, 58° - \frac{2}{3}\cot 58° \tan 32° - \frac{5}{3} \tan 13° \tan 37° \tan 45° \tan 53° \tan 77°$

(*ii*) $\left(\dfrac{\tan 20°}{\text{cosec}\, 70°}\right)^2 + \left(\dfrac{\cot 20°}{\sec 70°}\right)^2 + 2 \tan 15° \tan 45° \tan 75°.$

Solution.

(*i*) $\frac{2}{3}\text{cosec}^2\, 58° - \frac{2}{3}\cot 58° \tan 32° - \frac{5}{3}\tan 13° \tan 37° \tan 45° \tan 53° \tan 77°$

$= \frac{2}{3}\text{cosec}^2\, 58° - \frac{2}{3}\cot 58° \tan (90° - 58°)$

$- \frac{5}{3} \tan 13° \tan 37° . 1 . \tan (90° - 37°) \tan (90° - 13°)$

$= \frac{2}{3}\text{cosec}^2\, 58° - \frac{2}{3}\cot 58° \cot 58° - \frac{5}{3}\tan 13° \tan 37° \cot 37° \cot 13°$

$= \frac{2}{3}(\text{cosec}^2\, 58° - \cot^2 58°) - \frac{5}{3} \tan 13° . 1 . \cot 13°$

$= \frac{2}{3} . 1 - \frac{5}{3} . 1 = \frac{2}{3} - \frac{5}{3} = -1.$

(*ii*) $\left(\dfrac{\tan 20°}{\text{cosec}\, 70°}\right)^2 + \left(\dfrac{\cot 20°}{\sec 70°}\right)^2 + 2 \tan 15° \tan 45° \tan 75°$

$= \left(\dfrac{\tan 20°}{\text{cosec}\, (90° - 20°)}\right)^2 + \left(\dfrac{\cot 20°}{\sec (90° - 20°)}\right)^2 + 2 \tan 15° . 1 . \tan (90° - 15°)$

$= \left(\dfrac{\tan 20°}{\sec 20°}\right)^2 + \left(\dfrac{\cot 20°}{\text{cosec}\, 20°}\right)^2 + 2 \tan 15° \cot 15°$

$= \left(\dfrac{\sin 20°}{\cos 20°} . \cos 20°\right)^2 + \left(\dfrac{\cos 20°}{\sin 20°} . \sin 20°\right)^2 + 2 . 1$

$= \sin^2 20° + \cos^2 20° + 2$

$= 1 + 2 = 3.$

Example 6. *Express (sin 85° + cosec 85°) in terms of trigonometric ratios of angles between 0° and 45°.*

Solution. $\sin 85° + \text{cosec}\, 85° = \sin (90° - 5°) + \text{cosec}\, (90° - 5°)$

$= \cos 5° + \sec 5°.$

Example 7. *Prove the following :*

(i) $\dfrac{\sin(90° - A)\sin A}{\tan A} - 1 = -\sin^2 A$

(ii) $\dfrac{\sin A}{\sin(90° - A)} + \dfrac{\cos A}{\cos(90° - A)} = \sec A \operatorname{cosec} A.$

Solution.

(i) L.H.S. $= \dfrac{\sin(90° - A)\sin A}{\tan A} - 1 = \dfrac{\cos A \sin A}{\dfrac{\sin A}{\cos A}} - 1$

$= \cos^2 A - 1 = -(1 - \cos^2 A)$

$= -\sin^2 A =$ R.H.S.

(ii) L.H.S. $= \dfrac{\sin A}{\sin(90° - A)} + \dfrac{\cos A}{\cos(90° - A)}$

$= \dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A} = \dfrac{\sin^2 A + \cos^2 A}{\cos A \sin A}$

$= \dfrac{1}{\cos A \sin A} = \sec A \operatorname{cosec} A =$ R.H.S.

Example 8. *If $\sin 54° \operatorname{cosec}(90° - \theta) = 1$, find the value of θ, $0° < \theta < 90°$.*

Solution. Given $\sin 54° \operatorname{cosec}(90° - \theta) = 1$

$\Rightarrow \sin 54° \sec\theta = 1 \Rightarrow \sin 54° = \dfrac{1}{\sec\theta}$

$\Rightarrow \sin 54° = \cos\theta \Rightarrow \sin(90° - 36°) = \cos\theta$

$\Rightarrow \cos 36° = \cos\theta \Rightarrow \theta = 36°.$

Exercise 19.2

Without using trigonometric tables, evaluate the following (1 to 5) :

1. (i) $\dfrac{\cos 18°}{\sin 72°}$ (ii) $\dfrac{\tan 41°}{\cot 49°}$ (iii) $\dfrac{\operatorname{cosec} 17° 30'}{\sec 72° 30'}$.

2. (i) $\dfrac{\cot 40°}{\tan 50°} - \dfrac{1}{2}\left(\dfrac{\cos 35°}{\sin 55°}\right)$ (ii) $\left(\dfrac{\sin 49°}{\cos 41°}\right)^2 + \left(\dfrac{\cos 41°}{\sin 49°}\right)^2$

   (iii) $\dfrac{\sin 72°}{\cos 18°} - \dfrac{\sec 32°}{\operatorname{cosec} 58°}$ (2000)

   (iv) $\dfrac{\cos 75°}{\sin 15°} + \dfrac{\sin 12°}{\cos 78°} - \dfrac{\cos 18°}{\sin 72°}$ (2003)

   (v) $\dfrac{\sin 25°}{\sec 65°} + \dfrac{\cos 25°}{\operatorname{cosec} 65°}$. (2008)

3. (i) $\sin 62° - \cos 28°$ (ii) $\operatorname{cosec} 35° - \sec 55°$.

4. (i) $\cos^2 26° + \cos 64° \sin 26° + \dfrac{\tan 36°}{\cot 54°}$ (2012)

   (ii) $\dfrac{\sec 17°}{\operatorname{cosec} 73°} + \dfrac{\tan 68°}{\cot 22°} + \cos^2 44° + \cos^2 46°$. (2009)

5. (i) $\dfrac{\cos 58°}{\sin 32°} + \dfrac{\sin 22°}{\cos 68°} - \dfrac{\cos 38° \operatorname{cosec} 52°}{\tan 18° \tan 35° \tan 30° \tan 72° \tan 55°}$

   (ii) $\dfrac{\sec 29°}{\operatorname{cosec} 61°} + 2 \cot 8° \cot 17° \cot 45° \cot 73° \cot 82° - 3(\sin^2 38° + \sin^2 52°)$.

6. Express each of the following in terms of trigonometric ratios of angles between $0°$ to $45°$:

(i) $\tan 81° + \cos 72°$ (ii) $\cot 49° + \operatorname{cosec} 87°$.

Without using trigonometric tables, prove that (7 to 11) :

7. (i) $\sin^2 28° - \cos^2 62° = 0$ (ii) $\cos^2 25° + \cos^2 65° = 1$

(iii) $\operatorname{cosec}^2 67° - \tan^2 23° = 1$ (iv) $\sec^2 22° - \cot^2 68° = 1$.

8. (i) $\sin 63° \cos 27° + \cos 63° \sin 27° = 1$

(ii) $\sec 31° \sin 59° + \cos 31° \operatorname{cosec} 59° = 2$.

9. (i) $\sec 70° \sin 20° - \cos 20° \operatorname{cosec} 70° = 0$

(ii) $\sin^2 20° + \sin^2 70° - \tan^2 45° = 0$.

10. (i) $\dfrac{\cot 54°}{\tan 36°} + \dfrac{\tan 20°}{\cot 70°} - 2 = 0$

(ii) $\dfrac{\sin 50°}{\cos 40°} + \dfrac{\operatorname{cosec} 40°}{\sec 50°} - 4 \cos 50° \operatorname{cosec} 40° + 2 = 0$.

11. (i) $\dfrac{\cos 70°}{\sin 20°} + \dfrac{\cos 59°}{\sin 31°} - 8 \sin^2 30° = 0$

(ii) $\dfrac{\cos 80°}{\sin 10°} + \cos 59° \operatorname{cosec} 31° = 2$.

12. Without using trigonometrical tables, evaluate :

(i) $2\left(\dfrac{\tan 35°}{\cot 55°}\right)^2 + \left(\dfrac{\cot 55°}{\tan 35°}\right)^2 - 3\left(\dfrac{\sec 40°}{\operatorname{cosec} 50°}\right)$ (2011)

(ii) $\dfrac{\sin 35° \cos 55° + \cos 35° \sin 55°}{\operatorname{cosec}^2 10° - \tan^2 80°}$. (2010)

13. Prove the following :

(i) $\dfrac{\cos \theta}{\sin (90° - \theta)} + \dfrac{\sin \theta}{\cos (90° - \theta)} = 2$

(ii) $\cos \theta \sin (90° - \theta) + \sin \theta \cos (90° - \theta) = 1$

(iii) $\dfrac{\tan \theta}{\tan (90° - \theta)} + \dfrac{\sin (90° - \theta)}{\cos \theta} = \sec^2 \theta$.

14. Prove the following :

(i) $\dfrac{\cos (90° - A) \sin (90° - A)}{\tan (90° - A)} = 1 - \cos^2 A$

(ii) $\dfrac{\sin (90° - A)}{\operatorname{cosec} (90° - A)} + \dfrac{\cos (90° - A)}{\sec (90° - A)} = 1$.

15. Simplify the following :

(i) $\dfrac{\cos \theta}{\sin (90° - \theta)} + \dfrac{\cos (90° - \theta)}{\sec (90° - \theta)} - 3 \tan^2 30°$

(ii) $\dfrac{\operatorname{cosec} (90° - \theta) \sin (90° - \theta) \cot (90° - \theta)}{\cos (90° - \theta) \sec (90° - \theta) \tan \theta} + \dfrac{\cot \theta}{\tan (90° - \theta)}$.

16. Find the value of θ ($0° < \theta < 90°$) if :

(i) $\cos 63° \sec (90° - \theta) = 1$

(ii) $\tan 35° \cot (90° - \theta) = 1$.

CHAPTER TEST

Prove the following trigonometric identities (1 to 7) :

1. (*i*) $\frac{\cos A}{1-\sin A} + \frac{\cos A}{1+\sin A} = 2 \sec A$

(*ii*) $\frac{\cos A}{\text{cosec } A + 1} + \frac{\cos A}{\text{cosec } A - 1} = 2 \tan A.$

2. (*i*) $\frac{(\cos\theta - \sin\theta)(1+\tan\theta)}{2\cos^2\theta - 1} = \sec\theta$

(*ii*) $\frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta.$

3. (*i*) $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$

(*ii*) $\frac{\cot\theta}{\text{cosec}\,\theta + 1} + \frac{\text{cosec}\,\theta + 1}{\cot\theta} = 2\sec\theta.$

4. (*i*) $\sec^4 A(1 - \sin^4 A) - 2\tan^2 A = 1$

(*ii*) $\frac{1}{\sin A + \cos A + 1} + \frac{1}{\sin A + \cos A - 1} = \sec A + \text{cosec } A.$

Hint

(*i*) $\sec^4 A\,(1 - \sin^4 A) = \sec^4 A - \tan^4 A$
$= (\sec^2 A + \tan^2 A)(\sec^2 A - \tan^2 A)$
$= ((1 + \tan^2 A) + \tan^2 A).1$
$= 1 + 2\tan^2 A.$

5. (*i*) $\frac{1+\cos x}{1-\cos x} = \frac{\tan^2 x}{(\sec x - 1)^2}$

(*ii*) $\frac{\tan^2 x}{\tan^2 x - 1} + \frac{\text{cosec}^2 x}{\sec^2 x - \text{cosec}^2 x} = \frac{1}{\sin^2 x - \cos^2 x}.$

6. (*i*) $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = 1 + \sec A\,\text{cosec } A$

(*ii*) $(\sec A - \text{cosec } A)(1 + \tan A + \cot A) = \tan A \sec A - \cot A\,\text{cosec } A.$

7. $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0.$

Eliminate θ between the equations (8 to 9) :

8. $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$.

Hint

Add and subtract the given equations to find tan θ and sin θ. Take reciprocals to find cot θ and cosec θ and use $\text{cosec}^2\theta - \cot^2\theta = 1$.

9. $\cot\theta + \cos\theta = m$, $\cot\theta - \cos\theta = n$.

When 0° < θ < 90°, solve the following equations (10 to 13) :

10. (*i*) $3\tan^2 2\theta = 1$ (*ii*) $3\,\text{cosec}^2 3\theta = 4.$

11. (*i*) $\sec^2\theta - 2\tan\theta = 0$ (*ii*) $\tan^2\theta = 3(\sec\theta - 1).$

12. $2 \operatorname{cosec} \theta = 3 \sec^2 \theta$.

Hint

Given equation is $\frac{2}{\sin \theta} = \frac{3}{\cos^2 \theta} \Rightarrow 2 \cos^2 \theta = 3 \sin \theta$

$\Rightarrow 2(1 - \sin^2 \theta) = 3 \sin \theta \Rightarrow 2 \sin^2 \theta + 3 \sin \theta - 2 = 0.$

13. $3 \tan \theta + \cot \theta = 5 \operatorname{cosec} \theta$.

Hint

Given equation is $3.\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{5}{\sin \theta}$

$\Rightarrow 3 \sin^2 \theta + \cos^2 \theta = 5 \cos \theta \Rightarrow 3(1 - \cos^2 \theta) + \cos^2 \theta = 5 \cos \theta$

$\Rightarrow 2 \cos^2 \theta + 5 \cos \theta - 3 = 0.$

14. Without using trigonometrical tables, evaluate the following :

(*i*) $\sin^2 28° + \sin^2 62° - \tan^2 45°$

(*ii*) $\frac{2 \cos 27°}{\sin 63°} + \frac{\tan 27°}{\cot 63°} + \cos 0°$

(*iii*) $\cos 18° \sin 72° + \sin 18° \cos 72°$

(*iv*) $5 \sin 50° \sec 40° - 3 \cos 59° \operatorname{cosec} 31°$.

15. Prove the following :

(*i*) $\sin (90° - A) \cos (90° - A) = \frac{\tan A}{1 + \tan^2 A}$

(*ii*) $\frac{\cos (90° - \theta) \sec (90° - \theta) \tan \theta}{\operatorname{cosec} (90° - \theta) \sin (90° - \theta) \cot (90° - \theta)} + \frac{\tan (90° - \theta)}{\cot \theta} = 2.$

16. When $0° < A < 90°$, solve the following equations :

(*i*) $\sin 3A = \cos 2A$ (*ii*) $\tan 5A = \cot A$.

Hint

(*i*) $\sin 3A = \cos 2A \Rightarrow \sin 3A = \sin (90° - 2A) \Rightarrow 3A = 90° - 2A.$

17. Find the value of θ if

(*i*) $\sin (\theta + 36°) = \cos \theta$, where θ and $\theta + 36°$ are acute angles.

(*ii*) $\sec 4\theta = \operatorname{cosec} (\theta - 20°)$, where 4θ and $\theta - 20°$ are acute angles.