9 Matrices

9.1 MATRIX

*A rectangular arrangement of numbers, in the form of horizontal and vertical lines, is called a **matrix**.*

Horizontal lines are called ***rows*** and vertical lines are called ***columns***. Each number of a matrix is called its ***element***. The elements of a matrix are enclosed in brackets [ ].

Order of a matrix. If a matrix contains m rows and n columns, then it is called a matrix of *order* $m \times n$ (read as m by n).

A matrix of order $m \times n$ *has* mn *elements.*

An element appearing in the ith row and jth column of a matrix is called its *(i, j)th element* or the *(i, j)th entry*.

Notation. Matrices are usually denoted by capital letters, and the elements of a matrix by a small letter of the alphabet along with two suffixes, the first one indicating the number of row and the latter one, the number of the column in which the element appears. Thus, a matrix of order $m \times n$ may be written as

$$A = [a_{ij}]_{m \times n}.$$

For example :

(1) $\begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$ is a matrix of order 2 × 2. It has 4 elements. Here (1, 1)th element = 2, (1, 2)th element = 3, (2, 1)th element = 5 and (2, 2)th element = – 7.

(2) $A = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -5 & 11 \end{bmatrix}$ is matrix of order 2 × 3. It has 6 elements. Here $a_{22} = -5$, $a_{23} = 11$, $a_{12} = 3$ etc.

(3) $A = [2 \quad -1 \quad 0 \quad 6]$ is a matrix of order 1 × 4. It has four elements.

(4) $A = \begin{bmatrix} 2 & 9 \\ 5 & 8 \\ -1 & 0 \end{bmatrix}$ is a matrix of order 3 × 2. It has 6 elements.

9.1.1 Equal matrices

*Two matrices A and B are called **equal**, written as A = B, if and only if*

(i) A and B are of the same order i.e. number of rows in A = number of rows in B and number of columns in A = number of columns in B, and

(ii) *their corresponding elements are equal i.e. the entries of A and B in the same position are equal.*

Otherwise, the matrices are said to be ***unequal***, and we write $A \neq B$.

For example :

(1) The matrices $A = \begin{bmatrix} 2 & 5 \\ 7 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 7 & -4 \end{bmatrix}$ are equal, because both are of the same order 2×2 and their corresponding entries are equal.

(2) The matrices $A = \begin{bmatrix} 2 & 3 & 0 \\ 7 & -6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 7 & -6 & 5 \end{bmatrix}$ are not equal, because (1, 2)th entry of A ≠ (1, 2)th entry of B, even though both matrices A and B are of the same order 2×3.

(3) The matrices $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ are equal if and only if $x = 2$ and $y = -5$.

9.1.2 Some special types of matrices

1. **Row Matrix.** *A matrix having only one row is called a* ***row matrix.***

   For example, $[1 \quad -2 \quad 7 \quad 6]$ is a row matrix of order 1×4.

2. **Column Matrix.** *A matrix having only one column is called a* ***column matrix.***

   For example, $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ is a column matrix of order 3×1.

3. **Square Matrix.** *A matrix in which the number of rows equals the number of columns is called a* ***square matrix.*** Thus, a matrix of order $n \times n$ is called a square matrix of order n.

   For example, $\begin{bmatrix} 2 & 0 \\ -1 & -5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ are square matrices of orders 2 and 3 respectively.

   In a square matrix, the diagonal from the left top to the right bottom is called **principal** (or **leading**) **diagonal,** and all the elements in it are called **diagonal elements.**

   In the square matrix $\begin{bmatrix} 2 & 0 \\ -1 & -5 \end{bmatrix}$, the principal diagonal consists of 2, −5.

4. **Zero (or Null) Matrix.** *A matrix whose each element is zero is called a* ***zero*** (*or* ***null***) ***matrix.***

   For example, $[0 \quad 0]$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices of orders 1×2, 2×1, 2×2 and 2×3 respectively.

5. **Identity (or Unit) Matrix.** *A square matrix in which each diagonal element is 1 and all other elements are zero is called an* ***identity*** (*or* ***unit***) ***matrix.***

   For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of orders 2 and 3 respectively.

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Example 1. *If a matrix has 6 elements, what are the possible orders it can have ?*

Solution. Since all matrices of order 1×6, 6×1, 2×3 or 3×2 contain 6 elements, a matrix containing 6 elements can have any one of the following orders :

$$1 \times 6,\ 6 \times 1,\ 2 \times 3 \ \text{ or } \ 3 \times 2.$$

Example 2. *Construct a 2×2 matrix whose elements a_{ij} are given by $a_{ij} = i + j$.*

Solution. Given $a_{ij} = i + j$,

$\therefore \quad a_{11} = 1 + 1 = 2, \ a_{12} = 1 + 2 = 3,$

$\quad a_{21} = 2 + 1 = 3, \ a_{22} = 2 + 2 = 4.$

Hence, the required matrix $= \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$.

Example 3. *Find the values of a, b, c and d if* $\begin{bmatrix} a & -2 \\ b & 7 \end{bmatrix} = \begin{bmatrix} 2 & c \\ 3 & 2c+d \end{bmatrix}$.

Solution. Given $\begin{bmatrix} a & -2 \\ b & 7 \end{bmatrix} = \begin{bmatrix} 2 & c \\ 3 & 2c+d \end{bmatrix}$.

By difinition of equality of matrices, we get

$a = 2, \ b = 3, \ c = -2$ and $2c + d = 7$

$\Rightarrow \quad 2 \cdot (-2) + d = 7 \ \Rightarrow \ d = 7 + 4 = 11.$

Hence $a = 2, \ b = 3, \ c = -2$ and $d = 11$.

Example 4. *If* $\begin{bmatrix} x+3y & y \\ 7-x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$, *find the values of x and y.*

Solution. Given $\begin{bmatrix} x+3y & y \\ 7-x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$

By definition of equality of matrices, we get

$x + 3y = 4, \ y = -1$ and $7 - x = 0$

$\Rightarrow \quad x + 3y = 4, \ y = -1$ and $x = 7$.

Note that $x = 7$ and $y = -1$ satisfy $x + 3y = 4$.

Hence, $x = 7, y = -1$.

Example 5. *Find the values of x, y, a and b if* $\begin{bmatrix} x+y & a+b \\ a-b & 2x-3y \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & -5 \end{bmatrix}$.

Solution. By definition of equality of matrices, we get

$x + y = 5 \quad \ldots(i) \qquad a + b = -1 \quad \ldots(ii)$

$a - b = 3 \quad \ldots(iii) \qquad 2x - 3y = -5 \quad \ldots(iv)$

Adding (*ii*) and (*iii*), we get

$2a = 2 \ \Rightarrow \ a = 1.$

Putting $a = 1$ in (*ii*), we get

$1 + b = -1 \ \Rightarrow \ b = -2.$

To find x and y, multiplying (*i*) by 3, we get

$3x + 3y = 15 \quad \ldots(v)$

Adding (*iv*) and (*v*), we get

$5x = 10 \ \Rightarrow \ x = 2.$

Putting $x = 2$ in (*i*), we get

$2 + y = 5 \ \Rightarrow \ y = 5 - 2 = 3.$

Hence $x = 2, y = 3, a = 1$ and $b = -2$.

Exercise 9.1

1. Classify the following matrices :

(i) $\begin{bmatrix} 2 & -1 \\ 5 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -7 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 7 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 7 & 8 \\ -1 & \sqrt{2} & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2. (i) If a matrix has 4 elements, what are the possible orders it can have ?
(ii) If a matrix has 8 elements, what are the possible orders it can have ?

3. Construct a 2 × 2 matrix whose elements a_{ij} are given by
(i) $a_{ij} = 2i - j$ (ii) $a_{ij} = i.j$

4. Find the values of x and y if $\begin{bmatrix} 2x+y \\ 3x-2y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

5. Find the value of x if $\begin{bmatrix} 3x+y & -y \\ 2y-x & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$.

6. If $\begin{bmatrix} x+3 & 4 \\ y-4 & x+y \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 9 \end{bmatrix}$, find the values of x and y.

7. Find the values of x, y and z if $\begin{bmatrix} x+2 & 6 \\ 3 & 5z \end{bmatrix} = \begin{bmatrix} -5 & y^2+y \\ 3 & -20 \end{bmatrix}$.

8. Find the values of x, y, a and b if $\begin{bmatrix} x-2 & y \\ a+2b & 3a-b \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix}$.

9. Find the values of a, b, c and d if $\begin{bmatrix} a+b & 3 \\ 5+c & ab \end{bmatrix} = \begin{bmatrix} 6 & d \\ -1 & 8 \end{bmatrix}$.

10. Find the values of x, y, a and b if $\begin{bmatrix} 3x+4y & 2 & x-2y \\ a+b & 2a-b & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 5 & -5 & -1 \end{bmatrix}$.

9.2 OPERATIONS ON MATRICES

9.2.1 Addition of matrices

*If A and B are two matrices of the same order, we say that these are **compatible** (or **conformable**) **for addition**, and their sum A + B is the matrix obtained by adding the corresponding elements of A and B.*

For example :

(i) If $A = \begin{bmatrix} 2 & 3 \\ -5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 2 & -11 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 2+4 & 3+6 \\ -5+2 & 7+(-11) \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -3 & -4 \end{bmatrix}.$$

(*ii*) If $A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 29 & -4 \\ 5 & 11 & 3 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 5+(-1) & -1+29 & 4+(-4) \\ 2+5 & 7+11 & 0+3 \end{bmatrix} = \begin{bmatrix} 4 & 28 & 0 \\ 7 & 18 & 3 \end{bmatrix}.$$

9.2.2 Subtraction of matrices

If A and B are two matrices of the same order, we say that these are ***compatible for subtraction****, and their difference A – B is the matrix obtained by subtracting the elements of B from the corresponding elements of A.*

For example :

(*i*) If $A = \begin{bmatrix} 2 & 3 \\ -5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 2 & -11 \end{bmatrix}$, then

$$A - B = \begin{bmatrix} 2-4 & 3-6 \\ -5-2 & 7-(-11) \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -7 & 18 \end{bmatrix}.$$

(*ii*) If $A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 29 & -4 \\ 5 & 11 & 3 \end{bmatrix}$, then

$$A - B = \begin{bmatrix} 5-(-1) & -1-29 & 4-(-4) \\ 2-5 & 7-11 & 0-3 \end{bmatrix} = \begin{bmatrix} 6 & -30 & 8 \\ -3 & -4 & -3 \end{bmatrix}.$$

9.2.3 Multiplication of a matrix by a number

If k is any number and A is a matrix, then the matrix kA is obtained by multiplying each element of the matrix A by the number k.

For example :

(*i*) If $A = \begin{bmatrix} 2 & 3 \\ -5 & 7 \end{bmatrix}$, then

$$5A = \begin{bmatrix} 5\times 2 & 5\times 3 \\ 5\times(-5) & 5\times 7 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ -25 & 35 \end{bmatrix}.$$

(*ii*) If $A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 7 & 0 \end{bmatrix}$, then

$$(-3)\,A = \begin{bmatrix} (-3)\times 5 & (-3)\times(-1) & (-3)\times 4 \\ (-3)\times 2 & (-3)\times 7 & (-3)\times 0 \end{bmatrix} = \begin{bmatrix} -15 & 3 & -12 \\ -6 & -21 & 0 \end{bmatrix}.$$

Remarks

If A, B and C are matrices of the same order, then

- A + B = B + A (*addition of matrices is commutative*)
- (A + B) + C = A + (B + C) (*addition of matrices is associative*)
- A + O = A = O + A, where O is the zero matrix of order equal to order of A.
- A + (– A) = O, where O is zero matrix of order equal to order of A.

9.2.4 Solving matrix equations

To solve the matrix equation A + X = B where A, B and X (unknown) are matrices of the same order, we proceed in a manner similar to numbers.

On adding the matrix (–A) to both sides, we get (– A) + A + X = (– A) + B

⇒ ((– A) + A) + X = B – A

⇒ O + X = B – A

⇒ X = B – A, which is the required solution.

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Example 1. *If* $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & -1 \\ 4 & -7 \end{bmatrix}$, *find* $2A - 3B$.

Solution. $2A - 3B = 2\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix} + (-3)\begin{bmatrix} 0 & -1 \\ 4 & -7 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 10 \\ -4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -12 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 10+3 \\ -4+(-12) & 6+21 \end{bmatrix} = \begin{bmatrix} 2 & 13 \\ -16 & 27 \end{bmatrix}.$$

Example 2. *If* $2\begin{bmatrix} 3 & x \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ y & 2 \end{bmatrix} = \begin{bmatrix} z & -7 \\ 15 & 8 \end{bmatrix}$, *find the values of x, y and z.*

Solution. Given $2\begin{bmatrix} 3 & x \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ y & 2 \end{bmatrix} = \begin{bmatrix} z & -7 \\ 15 & 8 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 6 & 2x \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ 3y & 6 \end{bmatrix} = \begin{bmatrix} z & -7 \\ 15 & 8 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 6+3 & 2x+9 \\ 0+3y & 2+6 \end{bmatrix} = \begin{bmatrix} z & -7 \\ 15 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & 2x+9 \\ 3y & 8 \end{bmatrix} = \begin{bmatrix} z & -7 \\ 15 & 8 \end{bmatrix}$

$\Rightarrow \quad 9 = z, \quad 2x + 9 = -7, \; 3y = 15$

$\Rightarrow \quad z = 9, \quad 2x = -16, \; y = 5$

$\Rightarrow \quad z = 9, \quad x = -8, \; y = 5.$

Hence $x = -8$, $y = 5$ and $z = 9$.

Example 3. *Given* $A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, *find the matrix X such that* $A + X = 2B + C$. (2004)

Solution. Given $A + X = 2B + C \Rightarrow X = 2B + C - A$

$\Rightarrow \quad X = 2\begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -6 & 4 \\ 8 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6+1-2 & 4+0-(-1) \\ 8+0-2 & 0+2-0 \end{bmatrix} = \begin{bmatrix} -7 & 5 \\ 6 & 2 \end{bmatrix}.$$

Example 4. *If* $A = \begin{bmatrix} 3 & -4 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$, *find the matrix X if* $2A + 3X = 5B$.

Solution. Given $2A + 3X = 5B \Rightarrow 3X = 5B - 2A$

$\Rightarrow \quad 3X = 5\begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix} + (-2)\begin{bmatrix} 3 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ 15 & -5 \end{bmatrix} + \begin{bmatrix} -6 & 8 \\ 0 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 0+(-6) & 10+8 \\ 15+0 & -5+(-2) \end{bmatrix} = \begin{bmatrix} -6 & 18 \\ 15 & -7 \end{bmatrix}$$

$\Rightarrow \quad X = \frac{1}{3}\begin{bmatrix} -6 & 18 \\ 15 & -7 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 5 & -\frac{7}{3} \end{bmatrix}.$

Exercise 9.2

1. Given that M = $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ and N = $\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$, find M + 2N.

2. If A = $\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, find 2A − 3B.

3. If A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, compute 3A + 4B.

4. Given A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} -4 & -1 \\ -3 & -2 \end{bmatrix}$,

   (*i*) find the matrix 2A + B.

   (*ii*) find a matrix C such that C + B = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

5. If A = $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, B = $\begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$ and C = $\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$, find A + 2B − 3C.

6. If A = $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, find the matrix X if

   (*i*) 3A + X = B (*ii*) X − 3B = 2A.

7. Solve the matrix equation $\begin{bmatrix} 2 & 1 \\ 5 & 0 \end{bmatrix} - 3X = \begin{bmatrix} -7 & 4 \\ 2 & 6 \end{bmatrix}$.

8. If $\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} + 2M = 3\begin{bmatrix} 3 & 2 \\ 0 & -3 \end{bmatrix}$, find the matrix M. (*2008*)

9. Given A = $\begin{bmatrix} 2 & -6 \\ 2 & 0 \end{bmatrix}$, B = $\begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$ and C = $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

   Find the matrix X such that A + 2X = 2B + C. (*2013*)

10. Find X and Y if X + Y = $\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and X − Y = $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

11. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, find the values of x and y. (*2007*)

12. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 10 & 5 \end{bmatrix}$, find the values of x, y and z.

13. If $\begin{bmatrix} 5 & 2 \\ -1 & y+1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2x-1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ -7 & 2 \end{bmatrix}$, find the values of x and y.

14. If $\begin{bmatrix} a & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & b \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & c \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 7 & 3 \end{bmatrix}$, find the values of a, b and c.

15. If A = $\begin{bmatrix} 2 & a \\ -3 & 5 \end{bmatrix}$, B = $\begin{bmatrix} -2 & 3 \\ 7 & b \end{bmatrix}$, C = $\begin{bmatrix} c & 9 \\ -1 & -11 \end{bmatrix}$ and

    5A + 2B = C, find the values of a, b and c.

9.3 MULTIPLICATION OF MATRICES

Two matrices A and B are said to be ***compatible*** *(or* ***conformable****)* ***for the product AB*** *if and only if the number of columns in A is equal to the number of rows in B. If A is of order $m \times n$ and B is of order $n \times p$ then AB is of order $m \times p$, and is defined as $AB = [c_{ik}]_{m \times p}$ where*

$$(i, k)\text{th element of } AB = \begin{cases} \text{sum of the products of the elements of} \\ \text{the ith row of A with the corresponding} \\ \text{elements of the kth column of B.} \end{cases}$$

For example :

(*i*) Let $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ be two matrices.

Since A is of order 2×2 and B is of order 2×1, therefore, AB is defined and it is a matrix of order 2×1.

$$AB = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 3.4 + 5.7 \\ (-1).4 + 2.7 \end{bmatrix} = \begin{bmatrix} 47 \\ 10 \end{bmatrix}.$$

Note that B is of order 2×1 and A is of order 2×2, therefore, BA is not defined.

(*ii*) Let $A = \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 4 & 6 \end{bmatrix}$ be two matrices.

Since A is of order 2×2 and B is of order 2×2, therefore, AB is defined and it is a matrix of order 2×2.

$$AB = \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1.3 + 5.4 & 1.(-1) + 5.6 \\ 2.3 + 0.4 & 2.(-1) + 0.6 \end{bmatrix} = \begin{bmatrix} 23 & 29 \\ 6 & -2 \end{bmatrix}.$$

Also B is of order 2×2 and A is of order 2×2, therefore, BA is defined and it is a matrix of order 2×2.

$$BA = \begin{bmatrix} 3 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3.1 + (-1).2 & 3.5 + (-1).0 \\ 4.1 + 6.2 & 4.5 + 6.0 \end{bmatrix} = \begin{bmatrix} 1 & 15 \\ 16 & 20 \end{bmatrix}.$$

Observe that $AB \neq BA$.

Remark

From the above example it is clear that the ***multiplication of matrices is not commutative.***

ILLUSTRATIVE EXAMPLES

Example 1. *If* $A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} -3 & 4 \\ 6 & 0 \end{bmatrix}$, *verify that*

(*i*) $(AB)C = A(BC)$ (*ii*) $A(B + C) = AB + AC$.

Solution.

(*i*) $$AB = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2 + 3.5 & 1.4 + 3.1 \\ (-2).2 + 0.5 & (-2).4 + 0.1 \end{bmatrix} = \begin{bmatrix} 17 & 7 \\ -4 & -8 \end{bmatrix},$$

$$\therefore \quad (AB)C = \begin{bmatrix} 17 & 7 \\ -4 & -8 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 17.(-3)+7.6 & 17.4+7.0 \\ (-4).(-3)+(-8).6 & (-4).4+(-8).0 \end{bmatrix} = \begin{bmatrix} -9 & 68 \\ -36 & -16 \end{bmatrix} \quad \ldots(1)$$

$$BC = \begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 2.(-3)+4.6 & 2.4+4.0 \\ 5.(-3)+1.6 & 5.4+1.0 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ -9 & 20 \end{bmatrix},$$

$$\therefore \quad A(BC) = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 18 & 8 \\ -9 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 1.18+3.(-9) & 1.8+3.20 \\ (-2).18+0.(-9) & (-2).8+0.20 \end{bmatrix} = \begin{bmatrix} -9 & 68 \\ -36 & -16 \end{bmatrix} \quad \ldots(2)$$

From (1) and (2), we find that (AB)C = A(BC).

(*ii*) $$B + C = \begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-3) & 4+4 \\ 5+6 & 1+0 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 11 & 1 \end{bmatrix},$$

$$\therefore \quad A(B+C) = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 8 \\ 11 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.(-1)+3.11 & 1.8+3.1 \\ (-2).(-1)+0.11 & (-2).8+0.1 \end{bmatrix} = \begin{bmatrix} 32 & 11 \\ 2 & -16 \end{bmatrix} \quad \ldots(3)$$

$$AB = \begin{bmatrix} 17 & 7 \\ -4 & -8 \end{bmatrix} \quad \text{[from above]}$$

and $$AC = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.(-3)+3.6 & 1.4+3.0 \\ (-2).(-3)+0.6 & (-2).4+0.0 \end{bmatrix} = \begin{bmatrix} 15 & 4 \\ 6 & -8 \end{bmatrix},$$

$$\therefore \quad AB + AC = \begin{bmatrix} 17 & 7 \\ -4 & -8 \end{bmatrix} + \begin{bmatrix} 15 & 4 \\ 6 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 17+15 & 7+4 \\ -4+6 & -8+(-8) \end{bmatrix} = \begin{bmatrix} 32 & 11 \\ 2 & -16 \end{bmatrix} \quad \ldots(4)$$

From (3) and (4), we find that A(B + C) = AB + AC.

Remarks

- *Multiplication of matrices is associative i.e. if A, B and C are matrices conformable for multiplication, then (AB)C = A(BC).*
- *Multiplication of matrices is distributive with respect to addition i.e. if A, B and C are matrices conformable for the requisite addition and multiplication, then*

  A(B + C) = AB + AC and

  (A + B)C = AC + BC.

Example 2. *If* $A = \begin{bmatrix} 2 & -2 \\ 5 & -5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$, *compute AB and BA. What conclusions can you draw ?*

Solution. $AB = \begin{bmatrix} 2 & -2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2.3+(-2).3 & 2.4+(-2).4 \\ 5.3+(-5).3 & 5.4+(-5).4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$ and

$BA = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 5 & -5 \end{bmatrix}$

$= \begin{bmatrix} 3.2+4.5 & 3.(-2)+4.(-5) \\ 3.2+4.5 & 3.(-2)+4.(-5) \end{bmatrix} = \begin{bmatrix} 26 & -26 \\ 26 & -26 \end{bmatrix}.$

From the above products, it follows that

(*i*) AB = O, even though A ≠ O and B ≠ O.

(*ii*) AB ≠ BA *i.e.* multiplication of matrices is not commutative.

Example 3. *If* $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 3 & 3 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$, *show that AB = AC. What conclusion can you draw ?*

Solution. $AB = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1.4+(-1).3 & 1.5+(-1).3 \\ 2.4+(-2).3 & 2.5+(-2).3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ...(*i*)

and $AC = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1.2+(-1).1 & 1.7+(-1).5 \\ 2.2+(-2).1 & 2.7+(-2).5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ...(*ii*)

From (*i*) and (*ii*), we find that AB = AC.

From the above result, it follows that **cancellation law for the multiplication of matrices may not hold** *i.e.*

AB = AC may not imply B = C, A ≠ O.

Example 4. *Evaluate without using tables :*

$$\begin{bmatrix} 2\cos 60° & -2\sin 30° \\ -\tan 45° & \cos 0° \end{bmatrix} \begin{bmatrix} \cot 45° & \text{cosec } 30° \\ \sec 60° & \sin 90° \end{bmatrix}$$

Solution. $\begin{bmatrix} 2\cos 60° & -2\sin 30° \\ -\tan 45° & \cos 0° \end{bmatrix} \begin{bmatrix} \cot 45° & \text{cosec } 30° \\ \sec 60° & \sin 90° \end{bmatrix}$

$= \begin{bmatrix} 2.\frac{1}{2} & -2.\frac{1}{2} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (From trigonometry)

$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1.1+(-1).2 & 1.2+(-1).1 \\ (-1).1+1.2 & (-1).2+1.1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$

Example 5. *Given* $A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$ *and* $D = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, *find* $AB + 2C - 4D$. (2010)

Solution. $AB = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.6 + (-2).1 \\ (-1).6 + 4.1 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \end{bmatrix}$

$$\therefore \quad AB + 2C - 4D = \begin{bmatrix} 16 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 5 \end{bmatrix} + (-4)\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ -2 \end{bmatrix} + \begin{bmatrix} -8 \\ 10 \end{bmatrix} + \begin{bmatrix} -8 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} 16 + (-8) + (-8) \\ -2 + 10 + (-8) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = O.$$

Example 6. *Find x and y if* $\begin{bmatrix} x & 3x \\ y & 4y \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$. (2013, 02)

Solution. Given $\begin{bmatrix} x & 3x \\ y & 4y \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x.2 + 3x.1 \\ y.2 + 4y.1 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 5x \\ 6y \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

$\Rightarrow \quad 5x = 5$ and $6y = 12$

$\Rightarrow \quad x = 1$ and $y = 2$.

Hence $x = 1$ and $y = 2$.

Example 7. *Find x and y if* $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$.

Solution. Given $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 2.x + 3.y \\ (-1).x + 0.y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2x + 3y \\ -x \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$\Rightarrow \quad 2x + 3y = 7$ ...(*i*) and $-x = -2$ *i.e.* $x = 2$ ...(*ii*)

Putting $x = 2$ in (*i*), we get

$2 . 2 + 3y = 7 \quad \Rightarrow \quad 3y = 7 - 4 = 3 \quad \Rightarrow \quad y = 1.$

Hence $x = 2$ and $y = 1$.

Example 8. *Find x and y if* $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2x \\ 1 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 2 \\ y \end{bmatrix}$. (2003)

Solution. Given $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2x \\ 1 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 2 \\ y \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 3 . 2x + (-2) . 1 \\ (-1) . 2x + 4 . 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ 4y \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 6x - 2 \\ -2x + 4 \end{bmatrix} + \begin{bmatrix} -8 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ 4y \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} (6x-2)+(-8) \\ (-2x+4)+10 \end{bmatrix} = \begin{bmatrix} 8 \\ 4y \end{bmatrix} \Rightarrow \begin{bmatrix} 6x-10 \\ -2x+14 \end{bmatrix} = \begin{bmatrix} 8 \\ 4y \end{bmatrix}$$

$$\Rightarrow \begin{cases} 6x - 10 = 8 & \ldots(i) \\ -2x + 14 = 4y & \ldots(ii) \end{cases}$$

From (*i*), we get $6x = 18 \Rightarrow x = 3$.

Putting $x = 3$ in (*ii*), we get

$$-2.3 + 14 = 4y \quad \Rightarrow 4y = 8 \quad \Rightarrow y = 2.$$

Hence $x = 3$ and $y = 2$.

Example 9. *If* $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, *find the value of* $A^2 - 5A + 7I$, *where I is the unit matrix of order 2.* (2012)

Solution. $A^2 = AA = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3.3+1.(-1) & 3.1+1.2 \\ (-1).3+2.(-1) & (-1).1+2.2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix},$$

$$\therefore \; A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + (-5) \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8+(-15)+7 & 5+(-5)+0 \\ -5+5+0 & 3+(-10)+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $A^2 - 5A + 7I = O$.

Example 10. *If* $A = \begin{bmatrix} 2 & -3 \\ p & q \end{bmatrix}$, *find p and q so that* $A^2 = I$.

Solution. Given $A^2 = I \Rightarrow \begin{bmatrix} 2 & -3 \\ p & q \end{bmatrix} \begin{bmatrix} 2 & -3 \\ p & q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2.2+(-3).p & 2.(-3)+(-3).q \\ p.2+q.p & p.(-3)+q.q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 4 - 3p = 1 & \ldots(i) \\ -6 - 3q = 0 & \ldots(ii) \\ 2p + pq = 0 & \ldots(iii) \\ -3p + q^2 = 1 & \ldots(iv) \end{cases}$$

From (*i*) and (*ii*), we get $p = 1$, $q = -2$.

Note that these values of p and q satisfy (*iii*) and (*iv*).

Hence $p = 1$, $q = -2$.

Example 11. *Given* $A = \begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ *and* $BA = C^2$. *Find the values of p and q.* (2008)

Solution. $BA = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0.p+(-q).0 & 0.0+(-q).2 \\ 1.p+0.0 & 1.0+0.2 \end{bmatrix} = \begin{bmatrix} 0 & -2q \\ p & 0 \end{bmatrix}.$

$$C^2 = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2.2+(-2).2 & 2.(-2)+(-2).2 \\ 2.2+2.2 & 2.(-2)+2.2 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}.$$

Given BA = $C^2 \Rightarrow \begin{bmatrix} 0 & -2q \\ p & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$

$$\Rightarrow -2q = -8,\ p = 8 \Rightarrow q = 4,\ p = 8.$$

Hence $p = 8,\ q = 4$.

Example 12. *Given* $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $AB = A + B$, *find the values of* a, b *and* c.

Solution. $AB = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 3.a + 0.0 & 3.b + 0.c \\ 0.a + 4.0 & 0.b + 4.c \end{bmatrix}$

$$= \begin{bmatrix} 3a & 3b \\ 0 & 4c \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 3 + a & 0 + b \\ 0 + 0 & 4 + c \end{bmatrix}$$

$$= \begin{bmatrix} 3 + a & b \\ 0 & 4 + c \end{bmatrix}$$

Given AB = A + B $\Rightarrow \begin{bmatrix} 3a & 3b \\ 0 & 4c \end{bmatrix} = \begin{bmatrix} 3 + a & b \\ 0 & 4 + c \end{bmatrix}$

$$\Rightarrow \begin{cases} 3a = 3 + a \\ 3b = b \\ 4c = 4 + c \end{cases}$$

$\Rightarrow\ 2a = 3,\ \ 2b = 0,\ \ 3c = 4$

$\Rightarrow\ a = \frac{3}{2},\ \ b = 0,\ \ c = \frac{4}{3}.$

Hence, $a = \frac{3}{2},\ b = 0$ and $c = \frac{4}{3}$.

Example 13. *If* $A = \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}$ *and* $B = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$, *find matrix X such that* $2AX = B$.

Solution. Since 2AX = B and B is a 2 × 1 matrix

$\Rightarrow$ 2AX is a 2 × 1 matrix $\Rightarrow$ AX is a 2 × 1 matrix, but A is 2 × 2 matrix

$\Rightarrow$ X is a 2 × 1 matrix.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$, then 2AX = B

$$\Rightarrow\ 2\begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 6 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$$

$$\Rightarrow\ \begin{bmatrix} 4x + 6y \\ 0.x + (-4)y \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$$

$\Rightarrow\ 4x + 6y = -8 \quad \ldots(i) \quad$ and $\ -4y = 8 \Rightarrow y = -2.$

Putting $y = -2$ in (*i*), we get

$$4x + 6(-2) = -8 \Rightarrow 4x = 12 - 8 = 4 \Rightarrow x = 1.$$

$\therefore\ X = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$

Example 14. *If* $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & 3 \\ 1 & -11 \end{bmatrix}$, *find matrix B such that BA = C.*

Solution. Since BA = C and C is a 2 × 2 matrix

⇒ BA is a 2 × 2 matrix, but A is a 2 × 2 matrix

⇒ B is a 2 × 2 matirx.

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then BA = C

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+2b & -a+3b \\ c+2d & -c+3d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -11 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a + 2b = 2 & \ldots(i) \qquad -a + 3b = 3 \quad \ldots(ii) \\ c + 2d = 1 & \ldots(iii) \qquad -c + 3d = -11 \quad \ldots(iv) \end{cases}$$

Solving (*i*) and (*ii*), we get $a = 0, b = 1$.

Solving (*iii*) and (*iv*), we get $c = 5, d = -2$.

$$\therefore \quad B = \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix}.$$

Example 15. *If* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, *show that* $(aI + bA)^3 = a^3I + 3a^2bA$ *where I is the unit matrix of order 2.*

Solution. $aI + bA = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$,

$$\therefore \quad (aI + bA)^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a.a+b.0 & a.b+b.a \\ 0.a+a.0 & 0.b+a.a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix},$$

$$\therefore \quad (aI + bA)^3 = (aI + bA)^2\,(aI + bA) = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2.a+2ab.0 & a^2.b+2ab.a \\ 0.a+a^2.0 & 0.b+a^2.a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix} \quad \ldots(i)$$

$$a^3I + 3a^2bA = a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix} \quad \ldots(ii)$$

From (*i*) and (*ii*), we get

$$(aI + bA)^3 = a^3I + 3a^2bA.$$

Exercise 9.3

1. If $A = \begin{bmatrix} 3 & 5 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, is the product possible? Give a reason. If yes, find AB. *(2011)*

2. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$, find AB and BA. Is AB = BA ?

3. If $P = \begin{bmatrix} 4 & 6 \\ 2 & -8 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$, find 2PQ.

4. If $A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$, compute $(-A)^2$.

5. Given $A = \begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix}$, evaluate $A^2 - 4A$. *(2000)*

6. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find A(BA).

7. Given the matrices :

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} \text{ and } C = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix}.$$

Find the products of (*i*) ABC (*ii*) ACB and state whether they are equal.

8. Evaluate : $\begin{bmatrix} 4\sin 30^\circ & 2\cos 60^\circ \\ \sin 90^\circ & 2\cos 0^\circ \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$. *(2010)*

9. If $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -4 & -6 \end{bmatrix}$, find the matrix AB + BA.

10. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 1 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & -3 \\ 0 & 1 \end{bmatrix}$, find each of the following and state if they are equal :
    (*i*) CA + B (*ii*) A + CB.

11. If $A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$, find $2B - A^2$.

12. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$, compute
    (*i*) A(B + C) (*ii*) (B + C) A.

13. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, find the matrix C(B – A).

14. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$, find $A^2 + AB + B^2$. *(2007)*

15. Let $A = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 3 \\ 1 & -3 \end{bmatrix}$, find $A^2 - A + BC$. (2006)

16. If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, find A^2 and A^3. Also state which of these is equal to A.

17. If $X = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$, show that $6X - X^2 = 9I$ where I is the unit matrix.

18. Show that $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is a solution of the matrix equation $X^2 - 2X - 3I = O$ where I is the unit matrix of order 2.

19. Find the matrix X of order 2×2 which satisfies the equation :

$$\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix} + 2X = \begin{bmatrix} 1 & -5 \\ -4 & 6 \end{bmatrix}.$$

20. If $A = \begin{bmatrix} 1 & 1 \\ x & x \end{bmatrix}$, find the value of x so that $A^2 = O$.

21. If $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, find the value of x.

22. (*i*) Find x and y, if $\begin{bmatrix} -3 & 2 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ y \end{bmatrix}$. (2001)

(*ii*) Find x and y, if $\begin{bmatrix} 2x & x \\ y & 3y \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 9 \end{bmatrix}$. (2009)

23. Find the values of x and y if $\begin{bmatrix} x+y & y \\ 2x & x-y \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

24. Find x and y if $\begin{bmatrix} 2 & x \\ y & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix}$.

25. If $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} x & 0 \\ 9 & 0 \end{bmatrix}$, find the values of x and y.

26. If $\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, write down the values of a, b, c and d.

27. Find the value of x given that $A^2 = B$ where $A = \begin{bmatrix} 2 & 12 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & x \\ 0 & 1 \end{bmatrix}$. (2005)

28. If $A = \begin{bmatrix} 2 & x \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 36 \\ 0 & 1 \end{bmatrix}$, find the value of x, given that $A^2 = B$.

29. Evaluate x, y if $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2x \\ 1 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 4y \end{bmatrix}$.

30. If $\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} b & 11 \\ 4 & c \end{bmatrix}$, find a, b and c.

31. If A = $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & x \\ 0 & -\frac{1}{2} \end{bmatrix}$, find the value of x if AB = BA.

32. If A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, find x, y so that $A^2 = xA + yI$.

33. If P = $\begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$, Q = $\begin{bmatrix} 3 & x \\ y & 2 \end{bmatrix}$, find x, y such that PQ = O.

34. Let M × $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ = [1 2] where M is a matrix.

(*i*) State the order of the matrix M. (*ii*) Find the matrix M.

35. Given $\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} X = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, write :

(*i*) the order of the matrix X (*ii*) the matrix X. *(2012)*

36. Solve the matrix equation $\begin{bmatrix} 4 \\ 1 \end{bmatrix} X = \begin{bmatrix} -4 & 8 \\ -1 & 2 \end{bmatrix}$.

37. (*i*) If A = $\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$, find matrix C such that AC = B.

(*ii*) If A = $\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$ and B = [0 −3], find a matrix C such that CA = B.

38. If A = $\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$, find matrix B such that BA = I, where I is unity matrix of order 2.

39. If B = $\begin{bmatrix} -4 & 2 \\ 5 & -1 \end{bmatrix}$ and C = $\begin{bmatrix} 17 & -1 \\ 47 & -13 \end{bmatrix}$, find matrix A such that AB = C.

CHAPTER TEST

1. Find the values of a and b if $\begin{bmatrix} a+3 & b^2+2 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} 2a+1 & 3b \\ 0 & b^2-5b \end{bmatrix}$.

2. Find a, b, c and d if $3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & a+b \\ c+d & 3 \end{bmatrix} + \begin{bmatrix} a & 6 \\ -1 & 2d \end{bmatrix}$.

3. Find X if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.

4. Determine the matrices A and B when

$A + 2B = \begin{bmatrix} 1 & 2 \\ 6 & -3 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$.

5. (*i*) Find the matrix B if $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ and $A^2 = A + 2B$.

(*ii*) If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix}$, find A (4B – 3C).

6. If $A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, compute (AB)C and (CB)A.

Is (AB)C = (CB)A ?

7. If $A = \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, find each of the following and state it they are equal :

(*i*) (A + B) (A – B) (*ii*) $A^2 - B^2$.

8. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, find $A^2 - 5A - 14I$, where I is unit matrix of order 2×2.

9. If $A = \begin{bmatrix} 3 & 3 \\ p & q \end{bmatrix}$ and $A^2 = O$, find p and q.

10. If $A = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ x & y \end{bmatrix}$ and $A^2 = I$, find x and y.

11. If $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, find a, b, c and d.

12. Find a and b if $\begin{bmatrix} a-b & b-4 \\ b+4 & a-2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 14 & 0 \end{bmatrix}$.

13. If $A = \begin{bmatrix} \sec 60^\circ & \cos 90^\circ \\ -3\tan 45^\circ & \sin 90^\circ \end{bmatrix}$ and $B = \begin{bmatrix} 0 & \cot 45^\circ \\ -2 & 3\sin 90^\circ \end{bmatrix}$, find

(*i*) 2A – 3B (*ii*) A^2 (*iii*) BA.